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Renormalisation group analysis of the phase transition in the 2D Coulomb gas, Sine-Gordon theory and XY-model†

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Abstract. A systematic renormalisation group technique for studying the 2D sine-Gordon theory (Coulomb gas, XY model) near its phase transition is presented. The new results are (a) higher order terms in the flow equations, beyond those of Kosterlitz, give rise to a new universal quantity; (b) this in turn gives the universal form as well as the relative coefficient of the next-to-leading term in the correlation function of the XY model; (c) the free energy (1PI vacuum sum) is calculated after the singularity at $\beta^2 = 4\pi$ is treated; (d) vortices with multiple charges are shown to be irrelevant; (e) symmetry breaking fields are analysed systematically.

The main ideas are that the sine-Gordon theory can be defined as a double expansion in α(fugacity) and $\delta = \beta^2/8\pi - 1$ (distance from the critical temperature at $\alpha = 0$). Wave-function and coupling constant (α) renormalisations are necessary and sufficient, around $\beta^2 = 8\pi$ where $\cos\phi$ acquires dimension 2, for functions with elementary SG fields. This gives rise to renormalisation of β. The renormalisability is proved to the order we calculate in the context of the SG theory, and in general, by using the equivalence to the Thirring–Schwinger model. The renormalised β^2 plays a role analogous to the dimension in a ϕ^4 theory—8π being the critical dimension. $\beta^2 > 8\pi$ gives an infrared asymptotically free theory which leads to the well-known fixed line. The infrared properties are understood by analogy with the non-linear σ model.

1. Introduction

The Gaussian spin-wave approximation (Rice 1966, Kane and Kadanoff 1967, Wegner 1967, Berezinskii 1970, Zittartz 1976) gives the following description of the two-dimensional (2D) XY model (and systems, such as 2D He^4 films, believed to have the same critical behaviour): as required by the Mermin–Wagner (1966) theorem there is no long-range order (see also Coleman 1973) at all temperatures; the correlation length, ξ, is infinite and the spin correlation functions decay as power laws whose exponents vary continuously with temperature. For example, the function $\langle S_x(\boldsymbol{r})S_x(\boldsymbol{0})\rangle$ describing the correlations of one (say the x) component of the planar spins $\boldsymbol{S}(\boldsymbol{r})$ situated at lattice sites $\boldsymbol{r}$ behaves like $r^{-1/2\pi K}$ for large r, where the Hamiltonian $J\sum_{\langle r,r'\rangle}\boldsymbol{S}(\boldsymbol{r})\cdot\boldsymbol{S}(\boldsymbol{r}')$ includes only nearest-neighbour interactions and $K \equiv J/k_BT$. This picture is exact in the limit $T \to 0$, qualitatively correct for $k_BT \ll J$, and clearly wrong for $k_BT \gtrsim J$ where one expects a finite ξ and the associated exponential decay of correlations.

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Kosterlitz and Thouless (1973) first pointed out that inclusion of vortex excitations is essential because of their long-range interactions, and it corrects this defect in a simple way. The vortices carry integer vorticity and interact among themselves via the logarithmic 2D Coulomb interaction (see e.g. Onsager 1949). They can therefore be regarded as constituting a 2D Coulomb gas—a system of positive and negative integer charges interacting in two dimensions. At low temperatures (Kosterlitz and Thouless 1973, Kosterlitz 1974) the vortices are bound in pairs of zero vorticity and so affect the spin-wave description quantitatively but not qualitatively: the spin-wave exponent $(2\pi K)^{-1}$ is replaced by $(2\pi K_{\text{eff}})^{-1}$, where K_{eff} is a complicated (in general, incalculable) function of T which approaches K as $T \to 0$. At higher temperatures the binding of the vortices decreases until at $2\pi K_{\text{eff}} = 4$ a phase transition into a state composed of *unbound* vortices occurs. In this high-temperature phase ξ is finite and correlations decay exponentially. In the language of the 2D Coulomb gas this phase is composed of charged particles and is thus metallic; the screening length of the charges is simply given by ξ (Chui and Lee 1975). At $2\pi K_{\text{eff}} = 4$ the system undergoes a metal–insulator transition into an insulating phase composed of neutral molecules—bound pairs of charges—which provide no electrostatic screening; hence $\xi = \infty$.

Subsequent to their pioneering work, much effort has been devoted to imbedding Kosterlitz and Thouless' (1973) analysis in a more systematic framework. Most treatments start with the 'spin-wave plus Coulomb gas' (SWCG) approximation, sometimes referred to as the Villain model (Villain 1975). This model is specified by two parameters, the temperature T and a chemical potential μ for the Coulomb charges, both of which are determined by the XY coupling K. It has, however, proven useful to consider the SWCG in its own right, i.e. to regard T and μ as independent parameters. By imposing the appropriate relationship ($\mu = \mu(T)$) between μ and T one recovers an approximation to the XY model. This 'generalised' SWCG has a line of metal–insulator transitions: for each value of the chemical potential there is a critical temperature. Critical exponents such as $\eta \equiv (2\pi K_{\text{eff}}(T_c))^{-1}$ can then be defined for each point on the critical line, one point of which corresponds to the critical point of the XY model or a He^4 film.

Kosterlitz (1974) predicted, via renormalisation group (RG) methods, that η had the *universal* value of $\frac{1}{4}$ everywhere along the line. He also showed that as the line is approached from above, ξ diverges like $\exp[c(T-T_c)^{-1/2}]$, where c is a non-universal number. The first of these results implies $\eta = \frac{1}{4}$ for the 2D XY model and (Kosterlitz and Nelson 1977) a universal jump in the superfluid density of He^4 films at T_c. Kosterlitz's (1974) treatment left three obvious unresolved questions:

(i) His RG equations were derived only to lowest order in the couplings. It has thus far proven impossible to generate higher order RG equations with his methods; his equations are therefore not obviously the first step in a systematic scheme.

(ii) Kosterlitz and Thouless (1973) argued that higher-than-unit charges, being more strongly bound than unit charges at low temperature, remain bound up to temperatures T considerably greater than T_c, and so play no role in the transition. Kosterlitz (1974) thus ignored the higher charges; their irrelevance in the usual RG sense (Wilson and Kogut 1975) was not proven however.

(iii) The irrelevance of interactions such as spin-wave-vortex couplings was not (and has not been) established.

Subsequent (José *et al* 1978, Knops 1978) RG treatments of SWCG have reproduced Kosterlitz's findings, clarified the nature of the SWCG approximation to the 2D XY and other statistical mechanical models and, without carefully establishing points (ii) and (iii), provided strong evidence for them. A construction of a systematic RG procedure (point (i)) was proposed only recently (Wiegmann 1978) within the context of the Sine-Gordon (SG) theory, whose *exact* equivalence to the SWCG model is well-known (Coleman 1975, Chui and Lee 1975, Samuel 1978). Though Wiegmann realised the irrelevance of 'higher harmonics' $\cos n\beta\phi$ in the low-temperature XY phase, he still uses Kosterlitz's argument to discard multiply charged vortices. Using the transformations of Samuel (1978) one can rigorously show that these higher harmonics represent the higher vortices, and thus these vortices are not simply initially small, but are irrelevant (see below).

Wiegmann's (1978) work provides a most convincing confirmation of Kosterlitz's (1974) results, which have been challenged by Luther and Scalapino (1977). These authors, considering a slightly different 2D planar model, argue that ξ diverges as a power and that η is non-universal. It is possible to ascribe this discrepancy to non-universality among different 2D planar models, although more systematic analysis of the Luther–Scalapino (1977) programme indicates a stubborn drift towards the Kosterlitz (1974) results (P Pfeuty 1978 private communication).

Using a recursion relation (Wilson and Kogut 1974) approach, Wiegmann (1978) has shown how higher order terms enter the RG equations of the SG theory. The notorious awkwardness of recursion relations beyond lowest order leads him to incorrect equations in next-to-leading order.

However, in this paper we remedy this defect by exhibiting a systematic field-theoretic RG treatment for the SG theory. We verify all of Kosterlitz's (1974) and Wiegmann's (1978) results for the SWCG and in addition:

(i) calculate the correct next-to-leading order RG equations;

(ii) prove that a particular linear combination of the coefficients of these next-to-leading terms in universal;

(iii) show that the leading corrections to the scaling form of the spin correlation function are universal along the critical line and compute their universal coefficient.

Our starting point is the Euclidean SG Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\nabla\phi)^2 - [\alpha_0/(\beta_0^2 a^2)]\cos(\beta_0\phi) \tag{1.1}$$

for the single scalar field ϕ. The dimensionless coupling constants α_0 and β_0^2 are, respectively (Chui and Lee 1975, Minnhagen *et al* 1978), proportional to the fugacity $Z(\equiv e^{\mu/k_BT})$ and inverse temperature T of the SWCG model:

$$T = 2\pi\beta_0^{-2}$$
$$Z = \alpha_0/2\beta_0^2 \tag{1.2}$$

a is a short-distance (ultraviolet) cut-off. The critical line (Kosterlitz 1974) starts at the point $(\alpha_0, \beta_0^2) = (0, 8\pi)$.

To develop a field-theoretic RG treatment of the phase transition we must therefore understand how to renormalise (i.e. remove the infinities as $a \to 0$) the SG theory near $\beta_0^2 = 8\pi$. This idea seems, *a priori*, hopelessly irrational: for $\beta_0^2 \leqslant 8\pi$ Coleman (1975) has shown that normal ordering suffices to render the theory finite, but (e.g. Banks *et al* 1976) as β_0^2 approaches 8π the scale dimension of the operator $\cos\beta_0\phi$ approaches 2; at $\beta_0^2 = 8\pi$ the sum of the (individually finite) terms of the loop expansion at a given order in α_0 develop a logarithmic divergence as $a \to 0$; for $\beta^2 \geqslant 8\pi$, moreover, Coleman (1975) has proven the non-existence of the ground state of the normal ordered SG theory in the limit $a \to 0$.

As Banks *et al* (1976), Schroer and Truong (1977) and Wiegmann (1978) suggested, these facts point to the necessity for a second (viz wavefunction) renormalisation for $\beta_0^2 \geqslant 8\pi$. We assert that this additional subtraction permits a sensible definition of the SG theory as a double power series expansion in the (renormalised) coupling constants α and $\delta \equiv \beta^2/8\pi - 1$, even for $\delta > 0$. That is, *two* renormalisation constants suffice to remove all divergences order by order in the two expansion parameters.

We verify this proposition explicity to third order, where non-trivial cancellations involving overlapping divergences are already required. We do not have a direct proof to all orders within the context of the SG theory, but rather appeal to the well-known equivalence (Luther and Emery 1974, Coleman 1975, Luther 1976, Banks *et al* 1976, Frohlich and Seiler 1976) between the SG theory and the SU(2) Thirring Model. The renormalisability of the latter follows from power counting, thereby establishing our assertion.

Another difficulty was suggested by Schroer and Truong (1977). They argued that the theory becomes singular at $\beta^2 = 4\pi$, and hence even the high-temperature XY phase should be problematic. We show in § 10 that within our scheme—renormalisation about $\beta^2 = 8\pi$—the only place where the 4π divergence arises is in the free energy (sum of connected vacuum graphs) and that it can be handled systematically by a subtraction.

Thus, we are led to the conclusion that the renormalisation properties of the SG equation are analogous to those of the familiar ϕ^4 field theory in $4+\epsilon$ dimensions, with δ playing the role of ϵ and α the role of the renormalised ϕ^4 coupling constant: for $\delta < 0$ the theory (Coleman 1975) is super-renormalisable, for $\delta = 0$ it is renormalisable, and for $\delta > 0$ it is non-renormalisable but can be made finite order-by-order in powers of α and δ. We sharpen the analogy by showing that (like ϕ^4 theory for $\epsilon > 0$) the SG theory is *infrared* asymptotically free for $\delta > 0$. It is this property which makes the critical properties of the SWCG model calculable.

The present paper is planned as follows. In § 2 the model is defined, and the ultraviolet and infrared regularisations discussed. Section 3 explains the necessity of departing from a loop expansion in momentum space. Section 4 is a discussion of the renormalisation constants required to insure finiteness of the various correlation functions of interest and of symmetry properties. Section 5 describes in some detail the computation of the vertex functions and renormalisation constants of the SG theory to $O(\alpha^3)$. In § 6 we argue, by appealing to fermion models, that our renormalisation procedure works at all orders. The renormalisation constants are used in § 7 to derive renormalisation group equations (RGE) and flow equations, and to discuss the universality of the lowest order (Kosterlitz) version of the equations and the new universal quantity introduced by the higher order terms. Section 8 is devoted to the solutions of the flow equations, and § 9 uses these results to solve the RGE for the correlation function of the XY model. Section 10 deals with the free energy, and ϕ^2 operators. Finally § 11 is a discussion of the irrelevance of $\Delta S\eta\beta\phi$ operators (the higher harmonics) and § 12 contains a systematic treatment of the relevance of symmetry breaking fields h_p (José *et al* 1978). Seven appendices are devoted to various technical details.

2. Definition of the model

In performing a perturbation calculation with Lagrangian (1.1) in two dimensions, one faces infrared as well as ultraviolet divergences in each order. To deal with the former we introduce a mass term into the Lagrangian, thus breaking the symmetry of $\mathscr{L}$ under the discrete translation $\phi \to \phi + 2\pi n/\beta_0$ for integer n. We use the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m_0^2\phi^2 - (\alpha_0/a^2\beta_0^2)\cos\beta_0\phi \tag{2.1}$$

where the free propagator at low momentum is $G_0(p) = (p^2 + m_0^2)^{-1}$. We will carry out our calculations in the presence of m_0, and set $m_0 \to 0$ at the end. The volume is infinite throughout.

Our infrared treatment differs non-trivially from previous prescriptions, which typically preserve the $\phi \to \phi 2\pi n/\beta_0$ symmetry. Coleman (1975) introduces both an m_0 and a function, $f(x)$, of compact support multiplying the interaction; he then lets $m_0 \to 0$ in each order of perturbation theory holding $f(x)$ fixed. The equivalence among the SG theory, various fermionic models, and the SWCG is usually discussed with $m_0 = 0$ in a box of finite volume V. Our method has the advantage that the $m_0^2\phi^2$ term is unlike a finite volume, a local operator amenable to standard field-theoretic treatment.

It is, of course, normally assumed that, after perturbation theory is summed, finite results are obtained in the limit $f(x) \to 1$ or $V \to \infty$. We make the same claim for the limit $m_0 \to 0$ (see also appendices 5 and 6). Indeed, we shall see explicitly (§ 5) that while the renormalised vertex functions do not possess $m_0 \to 0$ limits in perturbation theory (see e.g. (5.10)) the solutions of our RGE do. Moreover, $m_0^2\phi^2$ will be shown to be a soft insertion and hence also a soft symmetry-breaker, which does not affect the renormalisation constants; the limit $m_0 \to 0$ will therefore coincide with the $V \to \infty$ or $f \to 1$ limits of earlier approaches. Our symmetry-breaking infrared procedure is closely analogous to treatments of the non-linear σ model near two dimensions (Brezin and Zinn-Justin 1976a,b, Brezin *et al* 1976, Amit *et al* 1978).

The ultraviolet regularisation is introduced by defining the free propagator in coordinate space:

$$G_0(x, a) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{ip\cdot y}}{p^2 + m_0^2}\bigg|_{y^2 = x^2 + a^2} = \frac{1}{2\pi} K_0[m_0(x^2 + a^2)^{1/2}] \tag{2.2}$$

whose Fourier transform is

$$G_0(p, a) = \frac{1}{p^2 + m_0^2} a(p^2 + m_0^2)^{1/2} K_1[a(p^2 + m_0^2)^{1/2}] \underset{p\to 0}{\sim} (p^2 + m_0^2)^{-1} \tag{2.3}$$

where K_0 and K_1 are the conventional Bessel functions (Gradshteyn and Ryzhik 1965).

Since repeated use will made of asymptotic properties of $G_0(x)$, we note that

$$G_0(x) \sim -\frac{1}{4\pi} \ln cm_0^2(x^2 + a^2) \qquad xm_0 \ll 1 \tag{2.4}$$

where $c = \frac{1}{4}e^{2\gamma}$; γ is Euler's constant. This regularisation is preferable to the usual sharp cut-off in momentum space since our calculations involve graphs with many loops and are most easily evaluated in coordinate space.

3. Beyond the loop expansion

By expanding the $\cos(\beta_0\phi)$ interaction of the SG theory in powers of ϕ and treating all resulting terms perturbatively, one generates graphs composed of vertices of all even orders and bare propagators $(p^2 + m_0^2)^{-1}$. One typically (Coleman 1975, Samuel 1978) classifies the graphs for a given Green function in terms of the number of internal momentum integrations (loops) and studies the superficial divergence of graphs by power counting.

Coleman (1978) noted that only graphs containing tadpoles (see e.g. figure 4) are ultraviolet-divergent, that each tadpole diverges logarithmically, and that these divergences can be eliminated by normal-ordering the cosine interaction. Normal-ordering removes (i.e. re-sums) the tadpoles, thereby renormalising α_0:

$$\alpha_0 a^{-2} \to \alpha_0 J \tag{3.1a}$$

$$J \equiv a^{-2} \exp[-\tfrac{1}{2}I(x=0)] = c(cm_0^2 a^2)^{\beta_0^2/8\pi - 1} m_0^2 \tag{3.1b}$$

where $I(x) \equiv \beta_0^2 G_0(x, a)$.

The theory is therefore renormalised by defining, for some arbitrary mass κ, a renormalised α via

$$a_0 = Z_\alpha \alpha \qquad Z_\alpha = (\kappa^2 a^2)^{-\beta_0^2/8\pi + 1} \tag{3.2}$$

where the (+1) in the last exponent compensates for the explicit a^2 in (1.1). The new coupling α satisfies the flow equation

$$\kappa \frac{\partial \alpha}{\partial \kappa}\bigg|_{\beta_0, \alpha_0, a} = \beta_\alpha \equiv -\alpha\kappa \frac{\partial \ln Z_\alpha}{\partial \kappa}\bigg|_{\beta_0, a} = \alpha(\beta_0^2/4\pi - 2). \tag{3.3}$$

The flow in the α, β_0 plane is depicted in figure 1. (The arrows indicate flow as the length scale increases.) This vertical flow pattern is at odds with Kosterlitz's (1974) results for the corresponding Coulomb plasma.

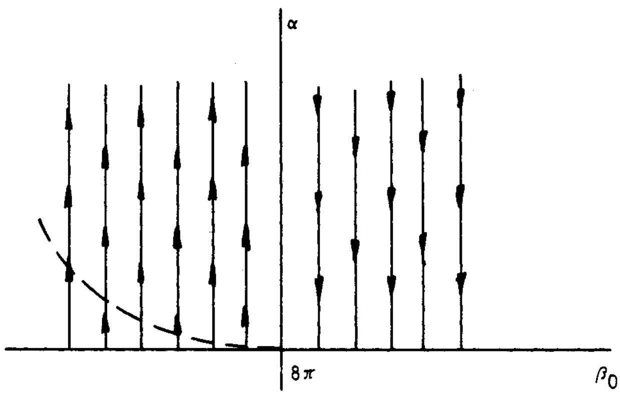

Figure 1. Flow patterns in the (α, β) plane with Coleman's renormalisation. The broken line is determined by a Ginzburg criterion. Below it the Coleman treatment holds. The arrows indicate the direction of the flow in the large-distance limit.

The source of the problem is the fact that as $\beta_0^2 \to 8\pi$, γ_α, the dimension of the coupling in front of the $\cos \beta_0\phi$, vanishes. This implies that the anomalous dimension (Wilson 1970, Amit 1978) of $\cos \beta_0\phi$ becomes equal to 2. Hence, it is a marginal operator—like a ϕ^4 operator at four dimensions (Wilson and Kogut 1974, Wegner 1974).

This analogy is helpful in understanding the left-hand part of the flow diagram, figure 1. The straight lines follow from a theory in which it is assumed that one is a finite distance within the region of super-renormalisability—$\beta_0^2 < 8\pi - \epsilon$, $\epsilon > 0$. In such a situation one cannot approach a scale-invariant (massless) theory because of infrared divergences (Symanzik 1973, 1975, see also discussion in Amit (1978, § 8–4)). To enter this region safely one must treat the cos $\beta_0\phi$ fully at $\beta_0^2 = 8\pi$, and then penetrate by an expansion into the super-renormalisable region as well as into the non-renormalisable one. (The fact that $\beta_0^2 > 8\pi$ is a non-renormalisable region was remarked by Schroer and Truong (1977).)

Of course, when $\beta_0^2 < 8\pi$ there is no scale invariance—the correlation length is finite. But the mass (the inverse correlation length, not to be confused with the externally introduced m_0) becomes smaller as $\beta_0^2 \to 8\pi$, and the region in α, in which the straight trajectories are valid, shrinks to zero rapidly. It will be confined below the broken line in figure 1. In other words, there is a Ginzburg criterion (Ginzburg 1960, Amit 1974) which delimits the validity of the calculation in the super-renormalisable regime.

Another way of making the point about the onset of marginality of cos $\beta_0\phi$, is to consider one of the finite graphs of the loop expansion such as graph (a) in figure 2. This graph is proportional to (see (2.2)):

$$\frac{1}{8\pi^2}\int d^2x K_0^2(m_0(x^2+a^2)^{1/2}) \sim \frac{1}{8\pi m_0^2} + A_1 a^2 \ln^2 m_0^2 a^2 + A_2 a^2 \ln m_0^2 a^2 + \dots .$$

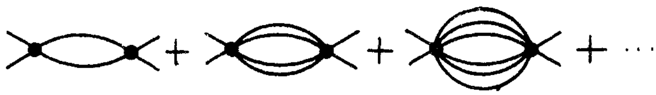

Figure 2. An example of an infinite set of graphs, all of second order in α which give a logarithmic divergence.

Clearly, as $a \to 0$, the integral tends to the finite limit $(8\pi m_0^2)^{-1}$. However, higher terms in the loop expansion of the same order in α_0—i.e. the remaining diagrams of figure 3 contribute terms of the form $a^2 \ln^{2n}(m_0 a)$. Since the dimension of $\cos(\beta_0\phi) \to 2$ as $\beta_0^2 \to 8\pi$ we might expect that for $\beta_0^2 \geq 8\pi$ the *sum* of these (individually convergent) graphs diverges.

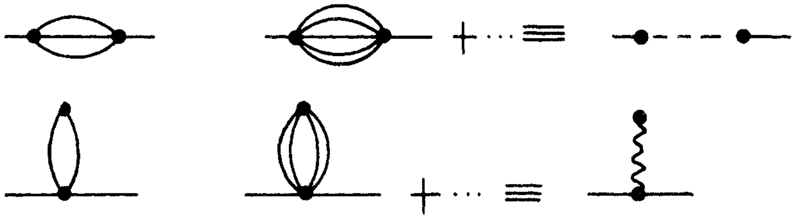

Figure 3. Graphs contributing to $\Gamma^{(2)}$ at second order in α. The RHS in the graphs defines the notation to be used below.

This expectation is correct. Consider, for example, the graphs (figure 3) for the 1PI two-point vertex $\Gamma^{(2)}(p)$ to second order in α. Their sum (Minnhagen *et al* 1978, Samuel 1978)

$$\Gamma_2^{(2)} = (\alpha_0 a^{-2})\beta_0^{-2}\int d^2x\{e^{ipx}[\sinh I(x) - I(x)] - [\cosh I(x) - 1]\} \tag{3.4}$$

diverges as $a \to 0$. To see this, note that since (2.4) implies, for small x,

$$\sinh I(x), \cosh I(x) \sim \tfrac{1}{2} e^{I(x)} = \sim\tfrac{1}{2}[cm_0^2(x^2+a^2)]^{-\beta_0^2/4\pi} \tag{3.5}$$

the ultraviolet-divergent part of (3.4) is

$$\Gamma_2^{(2)} \underset{a\to 0}{\sim} \tfrac{1}{2}(\alpha_0 a^{-2})^2 \beta_0^{-2} \int_0^\Delta d^2x (e^{ipx}-1)[cm_0^2(x^2+a^2)]^{-\beta_0^2/4\pi} \tag{3.6}$$

where Δ is an infinitesimal radius.

The a^{-4} in the prefactor is compensated by the renormalisation of α_0, equations (3.1) and (3.2). The integral itself is convergent as $a \to 0$ at $p = 0$ (Minnhagen *et al* 1978), but the coefficient of the p^2 term in (3.6) diverges logarithmically at $\beta_0^2 = 8\pi$ and even more strongly for $\beta_0^2 > 8\pi$. This suggests that wavefunction renormalisation be carried out.

4. Renormalisation Programme

4.1. Statement of the programme

In order to establish the flow equations for the temperature and the fugacity of the SWCG we study the renormalisation of $\Gamma^{(2)}(p)$ for the SG theory. We have seen that, calculating order by order in α_0, one encounters logarithmic divergences in $\Gamma^{(2)}$ at $\beta_0^2 = 8\pi$. We assert that all such infinities can be absorbed by two independent renormalisation constants, Z_ϕ and Z_α. That is, defining renormalised parameters α, β^2 and m, and a renormalised field ϕ_R via

$$\alpha_0 = Z_\alpha \alpha \tag{4.1a}$$

$$\beta_0^2 = Z_\phi^{-1}\beta^2 \tag{4.1b}$$

$$m_0^2 = Z_\phi^{-1} m^2 \tag{4.1c}$$

$$\phi^2 = Z_\phi \phi_R^2 \tag{4.1d}$$

we assert that Z_α and Z_ϕ can be chosen as functions of a to ensure that the renormalised two-point vertex

$$\Gamma_R^{(2)}(p, \alpha, \delta, m^2, \kappa) \equiv Z_\phi \Gamma^{(2)}(p, \alpha_0\delta_0, m_0^2, a) \tag{4.2}$$

is finite, order by order in a double expansion in α and $\delta \equiv \beta^2/8\pi - 1$, in the limit $a \to 0$. Here κ is a mass scale needed to define the renormalised theory, and note that $\beta_0^2\phi^2 = \beta^2\phi_R^2$ and $m_0^2\phi^2 = m^2\phi_R^2$, namely m_0 and β_0 undergo trivial renormalisation. It is convenient to obtain Z_α and Z_ϕ by requiring that the coefficients of p^2 and m^2 in $\Gamma^{(2)}$ be finite. Alternatively, one can impose the normalisation conditions

$$\Gamma_R^{(2)}(p^2 = 0, \alpha, \delta, m^2 = \kappa^2, \kappa) = \kappa^2 + \alpha c \kappa^2 \tag{4.3a}$$

$$\frac{d}{dp^2}\Gamma_R^{(2)}(p = 0, \alpha, \delta, m^2 = \kappa^2, \kappa) = 1 \tag{4.3b}$$

where c is defined in equation (2.4).

Anticipating that $m^2\phi^2$ is a 'soft' operator, we expect Z_ϕ and Z_α to be m-independent. In § 5 this claim is verified to third order in α and δ. These two variables are of the same order of magnitude in the vicinity of the critical lines (see also § 6).

General arguments supporting our assertions or renormalisability and m independence of the Z are given in § 6. These arguments indicate that renormalisation counter terms with the form of higher harmonics (i.e. $\cos(n\beta\phi)$) are not generated in the renormalisation process, even though they are allowed by the symmetry of the Lagrangian.

4.2. *Symmetry considerations*

The generating functional of the SG theory with $m_0 = 0$ (or partition function of the equivalent SWCG model) has the form

$$Z = \int D\phi \exp\left\{-\int d^2x[\tfrac{1}{2}(\nabla\phi)^2 - (\alpha_0/a^2\beta_0^2)\cos\beta_0\phi) + J(x)\phi(x)]\right\}. \quad (4.4)$$

This expression is formal, of course, and has to be supplemented by infrared and ultraviolet regularisation procedures before a meaningful perturbation theory can be extracted. Nevertheless, we can use it to investigate the symmetries of the theory. A useful check on our calculations follows from the invariance of $Z(J=0)$ under the transformation $\phi(x) \to \phi(x) + \pi/\beta_0$. This invariance implies that $Z(J=0)$ and all Green functions involving only derivatives of $\phi(x)$ are invariant under $\alpha_0 \to -\alpha_0$ in the limit $m_0 \to 0$ (i.e. depend only on α_0^2). But if the renormalisation constants are indeed m-independent, they must obey the symmetry.

This statement is connected (Coleman 1975) with the fact that the SWCG model—equivalent to the SG theory—has overall charge neutrality and so has equal numbers of positive and negative charges, and thus an even number.

The evenness of the Green functions implies that the renormalisation constants cannot contain odd powers of α. This in turn imposes constraints on the allowed α dependence of the flow functions, which are simply logarithmic derivatives of the renormalisation constants (see § 7.1).

The introduction of $m_0^2\phi^2$ into the Lagrangian destroys the $\alpha \to -\alpha$ symmetry. Odd powers of α will appear in the various functions. Nevertheless, if $m_0^2\phi^2$ is indeed a soft insertion, then the renormalisation constants will *not* be modified, and can be chosen to contain only even powers of α as before. This property will be checked explicitly in the calculation of the next subsection.

4.3. *Renormalisation of the free energy and correlations of ϕ^2*

The (dimensionless) free energy is defined as

$$F = -\ln Z(J=0) \quad (4.5)$$

where Z is the generating functional of the SG theory. Apart from the intrinsic interest of this function in context of the SG theory, it is the free energy of the SWCG and so related to the free energy of the XY model. In a ϕ^4 theory in four dimensions this function is not multiplicatively renormalisable, but needs three subtractions in addition to wavefunction, coupling constant, and mass renormalisations. Here we argue that, because $d = 2$, only two subtractions are required to eliminate all divergences from this function. This will be verified to $O(\alpha^2)$ in § 10.

The free energy has divergences of the form a^{-2}, and these are eliminated by the first subtraction. The additional logarithmic divergences are eliminated by the second one.

More explicitly, we show that one can construct a renormalised free energy via

$$F_R(\alpha, \delta, m^2, \kappa) = F_b(\alpha_0, \delta_0, m_0^2, a) - F_b(\alpha_0, \delta_0, \kappa^2 Z_\phi^{-1}, a) - (m_0^2 - \kappa^2 Z_\phi^{-1}) F_b'(\alpha_0, \delta_0, \kappa^2 Z_\phi^{-1}, a) \tag{4.6}$$

where F_b' is the derivative of F_B with respect to m_0^2.

The derivative of F with respect to m^2, $\langle\phi^2(x)\rangle$, needs only one subtraction in addition to a multiplicative renormalisation by Z_ϕ^{-1}, namely

$$F_R'(\alpha, \delta\, m^2 \kappa) = Z_\phi^{-1}(F_b'(\alpha_0, \delta_0, m_0^2, a) - F_b'(\alpha_0, \delta_0 \kappa^2 Z_\phi^{-1}, a)]. \tag{4.7}$$

The function $\langle\phi^2(x)\phi^2(y)\rangle$ needs no subtractions and is renormalised multiplicatively by Z_ϕ^{-2}.

That only Z_ϕ^{-1} is needed to renormalise ϕ^2 is (§ 5) a direct consequence of (4.1c).

5. Computation to third order

The first step in the computation of $\Gamma_b^{(2)}$ is the summation of the tadpoles, which (Coleman 1975) renormalises $\alpha_0 a^{-2}$ to $\alpha_0 J$, with J given by (3.1b). We represent this 'tadpole-renormalised' $\alpha_0 a^{-2}$, graphically, by a full square (figure 4). We shall use the graphical notation introduced in figure 3: the sum of even numbers of intermediate propagators is denoted by a wiggly line and stands for $[\cosh I(x) - 1]$; the sum of odd number of intermediate lines is represented by a broken line and stands for $[\sinh I(x) - I(x)]$.

All graphs contributing to $\Gamma_b^{(2)}$ up to third order in α_0 are shown in figure 5. Their sum can be written as

$$\begin{aligned}
&\Gamma_b^{(2)}(p; m_0, \alpha_0, \beta_0, a) \\
&\quad = p^2 + m_0^2 + \alpha_0 J - (\alpha_0 J)^2 \beta_0^{-2} \int d^2x \{e^{ipx}[\sinh I(x) - I(x)] - [\cosh I(x) - 1]\} \\
&\qquad + (\alpha_0 J)^3 \beta_0^{-4} \int dx\, dy\, \{e^{ipx}[\sinh I(x) - I(x)]\, e^{ipy}[\sinh I(y) - I(y)] \\
&\qquad - 2e^{ipx}[\sinh I(x) - I(x)][\cosh I(x) - 1] \\
&\qquad - e^{ipx} \sinh I(x)[\cosh I(y) - 1][\cosh I(x-y) - 1] \\
&\qquad + e^{ipx}[\cosh I(x) - 1] \sinh i(y) \sinh I(x-y) \\
&\qquad + [\cosh I(x) - 1][\cosh I(y) - 1] \\
&\qquad + \tfrac{1}{2}[\cosh I(x) - 1][\cosh I(y) - 1] \\
&\qquad + \tfrac{1}{2}[\cosh I(x) - 1][\cosh I(y) - 1][\cosh I(x-y) - 1] \\
&\qquad - \tfrac{1}{2} \sinh i(x) \sinh I(y) \sinh I(x-y)\}.
\end{aligned} \tag{5.1}$$

The (moment-independent) first order term in α_0 is proportional to J, which contributes factors of ln a as $\beta_0^2 \to 8\pi$. To third order in α_0 and δ_0 this term is

$$\Gamma_{b,1}^{(2)} = \alpha_0(cm_0^2)[1 + \delta_0 \ln(cm_0^2 a^2) + \tfrac{1}{2}\delta^2 \ln^2(cm_0^2 a^2)]. \tag{5.2}$$

The other source of divergences as $a \to 0$ is in the integrals. As was shown in § 3, the integral in the term quadratic in α_0 has a ln a divergence proportional to p^2. Its $p = 0$

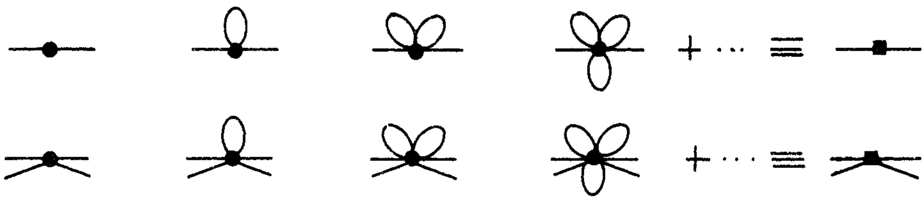

Figure 4. The two- and four-point interactions decorated by their tadpoles. The RHS defines our graphical notation for decorated interactions.

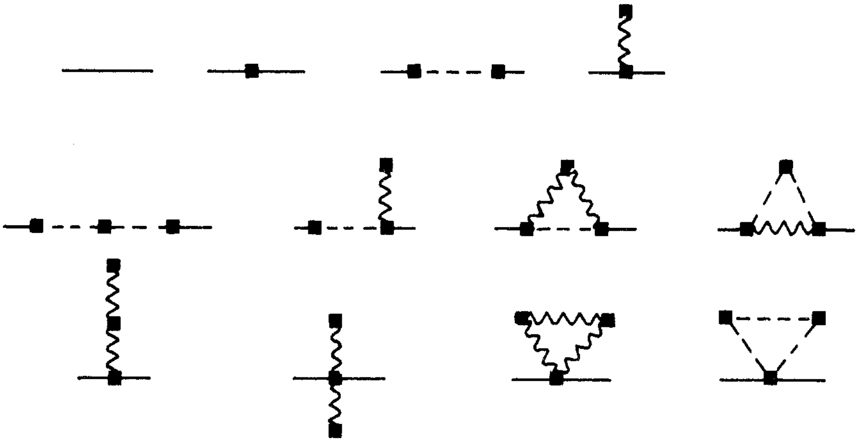

Figure 5. Graphs contributing to $\Gamma^{(2)}$, up to order α^3. The graphical notation is defined in figures 3 and 4.

term is finite. The divergent part of the $O(\alpha_0^2)$ integral is calculated in appendix 1. The resulting $O(\alpha_0^2)$ and $O(\alpha_0^2\delta_0)$ contributions are

$$\Gamma^{(2)}_{b,2} = -(\tfrac{1}{64})\alpha_0^2 p^2(1-\delta_0)[1+2\delta_0\ln(cm_0^2a^2)]$$
$$[\ln(cm_0^2a^2)-\delta_0\ln^2(cm_0^2a^2)-2\delta_0\ln(cm_0^2a^2)]. \qquad (5.3)$$

Both the finite, p-independent, part and the finite bit of the p^2 term of the integral threaten to give diverging contributions when multiplied by the prefactor $(\alpha_0 J)^2 \sim 1+2\delta_0\ln a$. These potential divergences disappear when α_0 is renormalised to remove the divergence in the first-order term, and so do not appear in (5.3) (see appendix 4). This type of cancellation is essential if the renormalisation constants are to preserve the $\alpha_0\to-\alpha_0$ symmetry.

Finally there is the $O(\alpha_0^3)$ term. Here one can set $\beta_0^2=8\pi$ everywhere. Only the first four terms in the integral (5.1) of the third-order term are momentum-dependent. We first argue that they contribute *no* singular term proportional to p^2.

One can readily show that their only potentially divergent p^2-term is:

$$[(\alpha_0 cm_0^2)^3/(8\pi)^2]p^2\int\cos^2\theta x^2\,\mathrm{d}^2x\,\mathrm{d}^2y\{\sinh I(x)[I(y)-\tfrac{1}{2}]$$
$$+\tfrac{1}{2}[\sinh I(x)\cosh I(y)\cosh I(x-y)-\cosh I(x)\sinh I(y)\sinh I(x-y)]\}$$

where θ is the angle between $\boldsymbol{p}$ and $\boldsymbol{x}$.

Here we have the first appearance of overlapping divergences: the x integration evidently produces a short-distance divergence whose coefficient depends on the integral over all values of y. The low values of y may lead to a $\ln^2$ divergence.

To extract the divergences note that the infinities can *only* arise from the following four regions of integration:

$$(1)\ \{|x|, |y|, |x-y|<\Delta\}; \qquad (2)\ \{|x|<\Delta; |y|, |x-y|>\Delta\};$$
$$(3)\ \{|y|<\Delta; |x|, |x-y|>\Delta\}; \quad (4)\ \{|x-y|<\Delta; |x|, |y|>\Delta\}.$$

In appendix 2 it is shown that the divergent contribution to the p^2 term vanishes as $\Delta \to 0$.

The p-independent part of the third-order term in (5.1) has contributions from all eight integrals. A bit of algebra reveals that the only singular part is:

$$\Gamma^{(2)}_{b,3} = [(\alpha_0 c m_0^2)^3/8^3\pi^2] \int d^2x\, d^2y\, \{\exp[I(x)+I(y)-I(x-y)] - \exp I(x) - \exp I(y)\}. \tag{5.4}$$

It also has overlapping divergences, and is analysed in regions:

$$(1)\ \{|x|<\delta; |y|>\Delta\}; \qquad (2)\ \{|x|<\Delta; |y|<\Delta\}; \qquad (3)\ \{|x|, |y|<\Delta\}. \tag{5.4a}$$

The details are left to appendix 3. The result is:

$$\Gamma^{(2)}_{b,3} = (\tfrac{1}{128})\alpha_0^3(cm_0^2)[\ln^2(cm_0^2a^2) + 5\ln(cm_0^2a^2)]. \tag{5.5}$$

Finite terms have again been discarded.

To sum up, the divergent part of $\Gamma^{(2)}_b$ to third order is

$$\begin{aligned}\Gamma^{(2)}_b = p^2 + m_0^2 &+ \alpha_0(cm_0^2)[1+\delta_0\ln(cm_0^2a^2) + \tfrac{1}{2}\delta_0^2\ln^2(cm_0^2a^2)] \\ &- (\tfrac{1}{64})\alpha^2p^2(1-\delta)[1-2\delta_0\ln(cm_0^2a^2)] \\ &\times[\ln(cm_0^2a^2) - \delta_0\ln^2(cm_0^2a^2) - 2\delta_0\ln(cm_0^2a^2)] \\ &+ (\tfrac{1}{128})\alpha_0^2(cm_0^2)[\ln^2(cm_0^2a^2) + 5\ln(cm_0^2a^2)].\end{aligned} \tag{5.6}$$

The next step is to expand Z_ϕ and Z_α in a double series in α and δ. Z_ϕ is expanded to third order, Z_α to second order, since according to (4.1a) $\alpha_0 = \alpha Z_\alpha$. These expansions are substituted into equations (4.1), which in turn are inserted in (5.6). In addition $\Gamma^{(2)}_b$ is multiplied by Z_ϕ, according to equation (4.2). Terms of $Z_\phi\Gamma^{(2)}$ are then rearranged by orders in the double expansion; the coefficients in Z_ϕ and Z_α are determined by the requirement that p^2 and m^2 have finite coefficients.

The result is:

$$Z_\phi = 1 + (\alpha^2/64)\ln\kappa^2a^2 - (\alpha^2\delta/64)[\ln^2\kappa^2a^2 + 3\ln\kappa^2a^2] + O(\alpha^4) \tag{5.7a}$$

$$Z_\alpha = 1 - \delta\ln\kappa^2a^2 + (\alpha^2/128)[\ln^2\kappa^2a^2 - 5\ln\kappa^2a^2] + (\delta^2/2)\ln^2\kappa^2a^2 + O(\alpha^3). \tag{5.7b}$$

The details are given in appendix 4.

To the order calculated here three assertions are confirmed by equations (5.7).

(a) That with two renormalisation constants, chosen specifically to renormalise α_0 and ϕ, the theory can be made finite.

(b) Z independent of m can be found.

(c) The $\alpha \to -\alpha$ symmetry is preserved by the Z.

As we discussed in the last section, properties (b) and (c) support our assertion that ϕ^2 is a soft insertion, and prove it to order α^3. A further check was made to test the softness of ϕ^2. The renormalised vertex $\Gamma_R^{(2,1)}(p_1, p_2; k=0)$, corresponding to $\langle\phi(x)\phi(y)\int\phi^2(z)\rangle$ is given by:

$$\Gamma_R^{(2,1)} = \frac{\partial}{\partial m^2}\Gamma_R^{(2)} = \frac{\partial}{\partial m^2} Z_\phi \Gamma_b^{(2)}. \tag{5.8}$$

Now if m_0^2 is renormalised by (4.1*c*) then

$$\Gamma_R^{(2,1)} = \frac{\partial}{\partial m_0^2}\Gamma^{(2)} = \Gamma_b^{(2,1)}. \tag{5.9}$$

Namely, $\Gamma^{(2,1)}$ requires no multiplicative factor. When the internal parameters are renormalised it should be finite. In fact, we have checked that to $O(\alpha^2)$ this is true of $\Gamma^{(2,1)}(p_1, p_2; k)$ even for $k \neq 0$.

Finally, we write the renormalised $\Gamma^{(2)}$ to $O(\alpha^3)$

$$\begin{aligned}\Gamma_R^{(2)} = p^2\{&1-(\alpha^2/64)\ln(cm^2/\kappa^2)-(\alpha^2/64)[\ln^2(cm^2/\kappa^2)-3\ln(cm^2/\kappa^2)]\}\\ &+m^2\{1+\alpha\delta c\ln(cm^2/\kappa^2)+(\alpha\delta^2/2)c\ln^2(cm^2/\kappa^2)\\ &+(\alpha^3c/128)[\ln^2(cm^2/\kappa^2)+5\ln(cm^2/\kappa^2)]\}.\end{aligned} \tag{5.10}$$

Note that for $p \neq 0$ the limit $m \to 0$ is divergent. This is completely analogous to the non-linear σ model in $2+\epsilon$ dimensions, as was remarked in the Introduction. The philosophy is that when the series is resumed, by the RGE for example, the limit can be safely taken.

It turns out that the infrared problems of perturbation theory occur in $\Gamma^{(2)}$, but not in Green functions of composite operators such as the *XY* spins or $\cos n\beta_0\phi$ (appendices 5 and 6). While $\Gamma^{(2)}$ is most handy for calculating the renormalisation of α and β, appendices 5 and 6 imply that the RC must be m-independent, or that ϕ^2 is soft. A similar situation arises in the non-linear σ model (Elitzur 1979).

6. General considerations on the renormalisability of the massive SG theory

Although in the previous section we justified our renormalisation procedure explicitly to third order, we have been unable to construct a direct proof of the renormalisability of SG theory to all orders. We shall instead construct a (non-rigorous) proof by appealing to the equivalence (Luther and Emery 1974) between the SG theory and various fermionic models.

Coleman (1975) showed the equivalence of the massive one-component Thirring model (TM) and the SG theory, order by order in perturbation theory. But the point $\beta^2 = 8\pi$ is not a simple point in the massive TM: it corresponds to $g = -\pi/2$, where $gJ_\mu J^\mu$ is the interaction.

On the other hand, Luther and Emery (1974) and Banks *et al* (1976) showed the equivalence between the SU(2)-massless TM and a theory of bosons consisting of a free field and a SG field, whose parameters α_0 and β_0 are linearly related: $\alpha_0 = -8(\beta_0^2/8\pi - 1)$.

More explicitly the SU(2) TM is described by a Lagrangian (Banks *et al* 1976):

$$\mathscr{L}=\mathrm{i}\bar{\psi}\partial\!\!\!/\psi-\tfrac{1}{2}g_{\mathrm{B}}^{0}J_{\mu}J^{\mu}-\tfrac{1}{2}g_{\mathrm{V}}^{0}\sum_{i=1}^{3}J_{\mu}^{(i)}J^{\mu(i)} \tag{6.1}$$

$$J_{\mu}=\sum_{a=1}^{2}\bar{\psi}^{a}\gamma_{\mu}\psi^{a}\qquad J_{\mu}^{(i)}=\sum_{a,b=1}^{2}\bar{\psi}^{a}\gamma^{\mu}\tfrac{1}{2}\tau_{ab}^{i}\psi^{b} \tag{6.2}$$

where a is the internal symmetry index of the spinor ψ and $\tau^{(i)}$ are Pauli matrices.

The corresponding theory of bosons is described by (Banks *et al* 1976):

$$\mathscr{L}=\tfrac{1}{2}\partial_{\mu}\theta^{(1)}\partial^{\mu}\theta^{(1)}+\tfrac{1}{2}\partial_{\mu}\theta^{(2)}\partial^{\mu}\theta^{(2)}-(g_{\mathrm{v}}^{0}\Lambda^{2}/2\pi^{2})\cos[\surd 8\pi(1+g_{\mathrm{v}}^{0}/2\pi)^{-1/2}\theta^{(2)}] \tag{6.3}$$

when one chooses the proper corresponding regularisations with Λ as the ultraviolet cut-off. The coupling g_{B}^{0} is absorbed into the normalisation of the free field.

One can see that there is a correspondence, in (6.3), between g_{v}^{0} and our parameters δ_0 and α_0:

$$\begin{aligned}\delta_0&=-g_0(1+g_0)^{-1}=-g_0+\mathrm{O}(g_0^2)\\ \alpha_0&=8g_0(1+g_0)^{-1}=8g_0+\mathrm{O}(g_0^2)\end{aligned} \tag{6.4}$$

where $g_0=g_{\mathrm{v}}^{0}/2\pi$. Therefore the double expansion in α_0 and δ_0 is an expansion in g_0. The exact relations (6.4) depend on the regularisation used to obtain the correspondence between (6.1) and (6.3). Only the first term in the expansion on the RHS of (6.4) is independent of it. Luther and Emery (1974), for example, obtain a different relation (see below).

Since the SU(2) TM is renormalisable by power counting (Mueller and Trueman 1971, Gomez and Lowenstein 1972) the corresponding SG theory must also be renormalisable, in the double expansion in α and δ along the line $\alpha=\alpha(g)$, $\delta=\delta(g)$, where $g=g(g_0, a\kappa)$ is the renormalised coupling of the SU(2) TM, and α and δ are the renormalised parameters of the corresponding SG theory. The line $\alpha=\alpha(\delta)$, obtained by eliminating g between the above relations, must correspond to a trajectory of the RGE of the SG theory passing through the point $\alpha=\delta=0$:

$$\alpha=-8\delta+\mathrm{O}(\delta^2). \tag{6.5}$$

As is shown independently in § 8, the linear term in (6.5) is universal (independent of regularisation). Relation (6.5) gives another special importance to the region in which α and δ are of the same order of magnitude.

So far we have argued in favour of renormalisation on one line in the (α, δ) plane. Let us modify the Lagrangian (6.1) by adding a term breaking the SU(2) symmetry, namely $-(\frac{1}{2})f_{\mathrm{v}}^{0}J_{\mu}^{(3)}J^{\mu(3)}$. Following in the footsteps of Banks *et al* (1976) one readily shows that this new Lagrangian corresponds to the boson theory

$$\mathscr{L}=\tfrac{1}{2}\partial_{\mu}\theta^{(1)}\partial^{\mu}\theta^{(1)}+\tfrac{1}{2}\partial_{\mu}\theta^{(2)}\partial^{\mu}\theta^{(2)}-(g_0\Lambda^{2}/\pi)\cos[\surd 8\pi(1+g_0+f_0)^{-1/2}\theta^{(2)}] \tag{6.6}$$

where $g_0=g_{\mathrm{v}}^{0}/2\pi$ and $f_0=f_{\mathrm{v}}^{0}/2\pi$. Hence we have obtained a SG theory with *two independent parameters*.

At lowest order the parameters α and δ are related to those of the fermion theory, defined by equation (6.7), via

$$\delta=-(g+f)\qquad \alpha=8g. \tag{6.7}$$

Using this correspondence one can show that the RGE, derived for the fermion theory, transform into those of the SG theory (see § 7).

The TM with broken SU(2) symmetry is, aside from normalisation conventions and relabellings of coupling constants, exactly Luther and Emery's (1974) 'backward scattering model'. Power counting arguments along the lines of Mueller and Trueman (1971) and Gomez and Lowenstein (1972) lead to the conclusion that this theory can also be renormalised, order by order in g and f by *two* renormalisation constants; this supports our assertion about the renormalisability of the SG theory.

In fact, we calculate with the massive rather than the massless SG theory. Frohlich and Seiler (1976) have proven the equivalence of the massive Thirring–Schwinger model and a massive SG theory. The mass of the SG theory is linear in e, the coupling of the gauge field to the fermion. This makes it plausible that an addition of a gauge field A_μ to the *massless* TM with broken SU(2) symmetry will make it equivalent to a massive SG theory near $\beta^2 = 8\pi$. Since the operator $\bar\psi\gamma_\mu\psi A^\mu$ is soft (it has dimension 1) it will not modify the renormalisation features of the theory. This supports our claim that $m_0^2\phi^2$ is soft when added to the massless SG theory.

7. RGE and universality

7.1. Flow equations for α and δ

From the renormalisation constants derived in § 5 one can extract the coefficients in the RGE obeyed by various Green's functions.

The β functions corresponding to the couplings α and δ are defined by

$$\beta_\alpha = \kappa(\partial\alpha/\partial\kappa)_\mathrm{b} = -\alpha\kappa(\partial \ln Z_\alpha/\partial\kappa)_\mathrm{b} \tag{7.1a}$$

$$\beta_\delta = \kappa(\partial\delta/\partial\kappa)_\mathrm{b} = (1+\delta)\kappa(\partial \ln Z_\phi/\partial\kappa)_\mathrm{b} \tag{7.1b}$$

where the subscript b means fixed bare parameters.

From the symmetry considerations discussed in § 4, it follows that β_α and β_δ contain only odd and even powers of α, respectively.

Using expressions (5.7) for Z_α and Z_ϕ we find

$$\beta_\alpha = 2\alpha\delta + \tfrac{5}{64}\alpha^3 \tag{7.2a}$$

$$\beta_\delta = \tfrac{1}{32}\alpha^2 - \tfrac{1}{16}\alpha^2\delta. \tag{7.2b}$$

The flow equations, under change ρ in the scale of length, are given by (Brézin *et al* 1976, Amit 1978)

$$\rho\frac{\mathrm{d}\alpha(\rho)}{\mathrm{d}\rho} = \beta_\alpha(\alpha(\rho), \delta(\rho)) \tag{7.3a}$$

$$\rho\frac{\mathrm{d}\delta(\rho)}{\mathrm{d}\rho} = \beta_\delta(\alpha(\rho), \delta(\rho)). \tag{7.3b}$$

As a check on the correspondences discussed in the last section one can derive the leading terms of (7.2) from the TM with broken SU(2) symmetry.

To lowest order one finds

$$\beta_g(g, f) = -2g(g+f) \qquad \beta_f(g, f) = 2gf. \tag{7.4}$$

Writing

$$\beta_\alpha = (\partial\alpha/\partial g)\beta_g + (\partial\alpha/\partial f)\beta_f$$

etc. one arrives, directly, at equations (7.2) at lowest order.

7.2. A new universal quantity

Let us express β_α and β_δ, derived in the previous subsection, in the form:

$$\beta_\alpha = 2\delta\alpha + A\alpha^3 \tag{7.5a}$$

$$\beta_\delta = \tfrac{1}{32}\alpha^2 + B\alpha^2\delta. \tag{7.5b}$$

The coefficients A and B separately are not universal numbers. They depend on the renormalisation procedure one employs. Nevertheless, we claim that, in addition to the coefficients of the lowest order terms, the combination $B+2A$ is a universal number. The argument is a generalisation of a method used by Gross (1976).

Under change in renormalisation there can be a change in the parameters α and δ to α' and δ' such that

$$\alpha' = \alpha + F\alpha\delta + \text{HOT} \tag{7.6a}$$

$$\delta' = \delta + G\alpha^2 + \text{HOT} \tag{7.6b}$$

with some finite numbers F and G. That these are the only possible changes follows from the relations

$$\alpha = \alpha_0 Z_\alpha^{-1} \to \alpha = \alpha_0(1 - A_1\delta + \ldots) \tag{7.7a}$$

$$\delta = \delta_0 Z_\phi + (Z_\phi - 1) \to \delta = \delta_0 + B_1\alpha^2 + \ldots . \tag{7.7b}$$

Under a finite renormalisation $A_1 \to \tilde{A}_1$, $B_1 \to \tilde{B}_1$, α and δ change into

$$\alpha' = \alpha + (A_1 - \tilde{A}_1)\alpha\delta \tag{7.8a}$$

$$\delta' = \delta + (\tilde{B}_1 - B_1)\alpha^2 \tag{7.8b}$$

which is just (7.6). In fact, if a *finite* renormalisation were to break the $\alpha \to -\alpha$ symmetry, the argument would be unaffected.

From (7.8) it follows that the new β functions $\beta'_{\alpha'}$ and $\beta'_{\delta'}$, corresponding to α' and δ' in the new renormalisation scheme, are:

$$\beta'_{\alpha'} = \beta_\alpha \frac{\partial\alpha'}{\partial\alpha} + \beta_\delta \frac{\partial\alpha'}{\partial\delta} = 2\alpha'\delta' + (\alpha')^3(A + F/32 - 2G) \tag{7.9a}$$

$$\beta'_{\delta'} = \beta_\alpha \frac{\partial\delta'}{\partial\alpha} + \beta_\delta \frac{\partial\delta'}{\partial\delta} = \frac{1}{32}(\alpha')^2 + (\alpha')^2\delta'(B - F/16 + 4G). \tag{7.9b}$$

Therefore the coefficients of the lowest order terms, as well as the combination

$$(B - F/16 + 4G) + 2(A + F/32 - 2G) = B + 2A$$

are independent of the renormalisation procedure. The role of this new universal number will be discussed in §§ 8 and 9.

7.3. Renormalisation group equations

A function which is renormalised via

$$G_R(p^2; \alpha, \delta, m^2, \kappa) = ZG_b(p^2; \alpha_0, \delta_0, m_0^2) \tag{7.10}$$

satisfies a RGE (Brezin and Zinn-Justin 1976a, b, Amit 1978)

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta_\alpha(\alpha, \delta)\frac{\partial}{\partial\alpha} + \beta_\delta(\alpha, \delta)\frac{\partial}{\partial\delta} + \gamma_\phi(\alpha, \delta)m^2\frac{\partial}{\partial m^2} + \gamma(\alpha, \delta)\right)G_R(p; \alpha, \delta, m^2, \kappa) = 0 \tag{7.11}$$

where we defined

$$\gamma_\phi = \kappa\frac{\partial \ln Z_\phi}{\partial\kappa}\bigg|_b \tag{7.12}$$

$$\gamma = -\kappa\frac{\partial \ln Z}{\partial\kappa}\bigg|_b. \tag{7.13}$$

If G stands for the usual N-point vertex function $\Gamma^{(N)}$ of the SG theory then $Z = Z_\phi^{N/2}$, $\gamma = -\frac{1}{2}N\gamma_\phi$. For functions G containing composite operators, γ is determined by the appropriate Z, characterising the operators.

If the function considered needs subtractions in order to be renormalised, it satisfies an inhomogeneous RGE. Such equations will be considered in § 10.

8. Solution of the RGE

8.1. Discussion of the flow equations

Using expressions (7.3) for the β functions we write the flow equations (7.1) for the effective coupling constants $\alpha(\rho)$ and $\delta(\rho)$ in the form

$$\rho\frac{dy}{d\rho} = xy + A_1y^3 \tag{8.1a}$$

$$\rho\frac{dx}{d\rho} = y^2 + B_1y^2x. \tag{8.1b}$$

Here $A_1 = 16A = 5/4$, $B_1 = 16B = -1$, $B_1 + 2A_1$ is a universal number (3/2) whose role in the solution of (8.1) will emerge momentarily, and $y \equiv \alpha/4$, $x \equiv 2\delta$. To lowest order (8.1) are just the flow equations derived by Kosterlitz (1974) for the SWCG (recall equation (1.2)) and reproduced by José *et al* (1978).

The solution to Kosterlitz's equations is well known. The flow diagram (figure 6) is not modified qualitatively by the third-order terms. It consists of three regions, denoted in the figure by I, II and III. The flow along the trajectories takes place as the value of ρ changes from one to zero; the arrows point in the direction of this flow.

One can trace the behaviour of the couplings in the infrared ($\rho \to 0$) limit. In region I, which corresponds to the insulating (low-temperature) phase of the Coulomb gas, the theory is IR asymptotically free ($\alpha \to 0$). The separatrix between region I and II is the phase transition (critical) line, any point of which flows into the origin $y = x = 0$ ($\alpha = \delta = 0$) as $\rho \to 0$. Its equation $y = y(x)$ is not universal except in lowest order where $y(x) = x$. The whole line $y = 0$, $x \geqslant 0$ ($\delta \geqslant 0$) is a line of infrared *stable* fixed points.

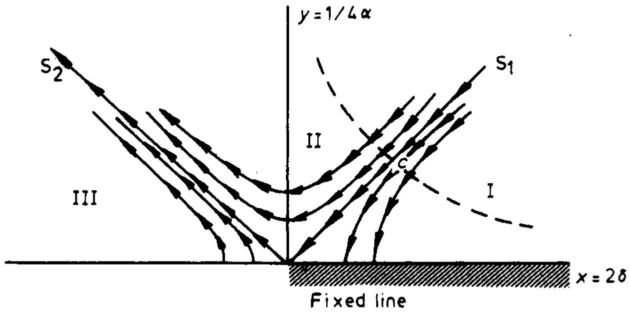

Figure 6. The celebrated Kosterlitz flow scheme for the Coulomb gas. There are three qualitatively different regions, separated by two separatrices S_1, S_2.

Regions II and III constitute the metallic (high-temperature) phase of the SWCG. In region II the effective coupling constants run towards high values and soon reach the region where our approximations (small x, y) are no longer valid. This takes place at short as well as long length scales.

Region III (which differes substantially from region II in the ultraviolet limit) is also characterised by the running away of the effective couplings towards higher values at large distances. Here the theory is asymptotically free in the *ultraviolet.* The trajectories start perpendicularly from the x axis, as is implied by Coleman's (1975) approach (see e.g. § 3).

The XY model has only one parameter $K = J/k_B T$, and therefore corresponds to a line, namely (Kosterlitz 1974, José *et al* 1978),

$$y = c \exp[-\tfrac{1}{4}\ln(8e^{2\gamma})x] \tag{8.2}$$

in the XY plane; this line is drawn broken in figure 6.

Following Kosterlitz, we identify the critical point (point C in figure 6) of the XY model as the intersection of this line with the critical line $y = y(x)$. This point is, of course, not universal. Starting at the critical point of the XY model, one is driven along the separatrix to the trivial fixed point $\alpha = \delta = 0$.

We now investigate the influence of the higher corrections to Kosterlitz's flow equations. It is simple to show that the invariant of these equations to third order is

$$f(x, y) = x^2 - y^2 - 2A_1xy^2 - 3^{-1}(4A_1 + 2B_1)x^3. \tag{8.3}$$

The RG flow lines are then given by $f(x, y) = C$ for different values of the constant C. The critical line (separatrix) corresponds to $C = 0$, i.e.

$$y^2 = x^2 + 2(A_1 - B_1)x^3/3 + O(x^4). \tag{8.4}$$

This equation is not universal. However, solving for $x(\rho)$ one finds

$$\rho\frac{dx}{d\rho} = y^2(1 + B_1x) = x^2 + \tfrac{1}{3}(B_1 + 2A_1)x^3. \tag{8.5}$$

In this equation the universal number $B_1 + 2A_1 = 16(B + 2A)$ enters. It can be easily solved to give

$$\ln \rho = -\frac{1}{x(\rho)} + \frac{1}{x_i} + \frac{1}{3}(B_1 + 2A_1) \ln \frac{1/x(\rho) + (B_1 + 2A_1)/3}{1/x_i + (B_1 + 2A_1)/3} \tag{8.6}$$

Since $x(\rho)$ vanishes in the limit of small ρ one obtains, as $\rho \to 0$,

$$\ln \rho \to -1/x(\rho) - \tfrac{1}{3}(B_1 + 2A_1) \ln x(\rho). \tag{8.7}$$

Taking as a first approximation $x(\rho) = |\ln \rho|^{-1}$, we iterate (8.7) to find

$$x(\rho) \sim 1/|\ln \rho| - \tfrac{1}{3}(B_1 + 2A_1) \ln|\ln \rho|/|\ln \rho|^2. \tag{8.8}$$

Recall $B_1 + 2A_1 = 3/2$. This result is rather similar to results found by Symanzik (1973b, equation 3.17) for a ϕ^4 theory.

It is not difficult to construct solutions of the flow equations in other regions by the same method. Indeed, Kosterlitz's (lowest order) can be solved analytically in the whole plane (see e.g. the solution in region I exhibited in § 9).

In region II one can label an arbitrary flow by y_0—its intersection with the line $x = 0$. Use of (8.3) then facilitates the integration of (8.1*b*), whereupon:

$$\begin{aligned} \ln \rho = {} & -\pi/y_0 - 1/x + 1/x_i + \tfrac{1}{3}(B_1 + 2A_1) y_0^2 [(x^2 + y_0^2)^{-1} - (x_i^2 + y_0^2)^{-1}] \\ & - \tfrac{1}{3}(B_1 + 2A_1) \ln[(x^2 + y_0^2)(x_i^2 + y_0^2)^{-1}]. \end{aligned} \tag{8.9}$$

8.2. *Solution of the* RGE *using effective coupling constants*

In § 7 we established the RGE (7.7), satisfied by a function G multiplicatively renormalised by a renormalisation constant Z. The solution of this equation is (Amit 1978, § 10.13):

$$G(p; \alpha, \delta, m^2, \kappa) = \rho^D \exp\left(-\int_\rho^1 \gamma[\alpha(x), \delta(x)] \frac{dx}{x}\right) G(\rho^{-1}p; \alpha(\rho), \delta(\rho), m^2(\rho), \kappa). \tag{8.10}$$

Here D is the physical dimension of G in units of mass and $m^2(\rho)$ is given by

$$m^2(\rho) = m^2 \exp\left(\int_\rho^1 \frac{dx}{x} [2 - \gamma_\phi(\alpha(x), \delta(x))]\right) \tag{8.11}$$

with $m^2(1) = m^2$.

Taking the limit $m \to 0$ in the full solution (which is impossible in perturbation theory) one finds

$$G(p; \alpha, \delta, 0, \kappa) = \rho^D \exp\left(\int_\rho^1 \gamma[\alpha(x), \delta(x)] \frac{dx}{x}\right) G(p^{-1}p; \alpha(\rho), \delta(\rho), 0, \kappa). \tag{8.12}$$

8.3. *The behaviour of the correlation length*

In region II of the phase diagram the correlation length of the SWCG (or SG or XY model) is identified as the inverse value of ρ for which $\alpha(\rho)$ and $\delta(\rho)$ become of order unity. If one assumes that a given correlation function G is free of singularities for these values of $\alpha(\rho)$ and $\delta(\rho)$ (i.e. at high temperatures) then according to (8.12) $G(p; \alpha, \delta, 0, \kappa) \sim \xi^D \exp \int_\xi^1 \gamma[\alpha(x), \delta(x)]\, dx/x$. We make the further assumption that no singularity is approached in the vicinity of the (high-temperature) separatrix between regions II and III. It then follows from (8.9) with $x(\rho) = 1$ that:

$$\ln \xi = \frac{\pi}{y_0} - \left(1 + \frac{1}{x_i}\right) + \frac{B_1 + 2A_1}{6} y_0^2 \left(1 - \frac{1}{x_i^2}\right) - \ln x_i^2 + O(y_0^4). \tag{8.13}$$

Now

$$y_0^2 = bt + O(t^2) \tag{8.14}$$

where b is some non-universal constant and $t \equiv (T-T_c)/T_c$ is the deviation of the SWCG temperature from its critical values. Hence

$$\xi \propto \exp[\pi(ct)^{-1/2}][1+O(t)] \tag{8.15}$$

where the coefficient of the linear power of t is not universal. It thus follows that the higher-order corrections to the flow equation do not affect the behaviour of the correlation length in any interesting way. The situation is different with regard to the correlation function.

Using the same logic one can easily convince oneself that in region I of figure 6 ξ is infinite—the theory is massless.

8.4. Remnants of asymptotic freedom

In region I of figure 6—the low-temperature XY region—the theory is (see (8.8)) IR asymptotically free, as it is along the separatrix. It is well known (Symanzik 1973b) that one cannot simply substitute the limiting values (i.e. zero) of the coupling constants in equations like (8.12).

If the coupling constant starts as $(\ln\rho)^{-1}$, and if an anomalous dimension function $\gamma(x)$ (say, of one variable) behaves like

$$\gamma(x) \underset{x\to 0}{\sim} \gamma_0 + \gamma_1 x^s$$

then, in the asymptotic region, $\rho \to 0$, equations like (8.12) will have a term of the form

$$\exp\left(\int_\rho^1 \frac{dz}{z}[\gamma_0+\gamma_1(\ln z)^{-s}]\right) \sim \rho^{-\gamma_0}\exp\left(\frac{\gamma_1}{s-1}(\ln\rho)^{1-s}\right). \tag{8.16}$$

For any $s>1$, as $\rho\to 0$, the exponential factor tends to a constant. For $s=1$ the prefactor behaves as $\rho^{-\gamma_0}(\ln\rho)^{\gamma_1}$. Thus all terms in the γ of higher order than 1, in a coupling constant which vanishes asymptotically, are unimportant. This fact will be used repeatedly in the following sections.

9. The *XY* correlation function

9.1. General considerations

In order to find the behaviour of the couplings α and β it was advantageous to consider the Green functions of SG fields. Clearly, a Green function of any set of more complex operators will have to have its α and β renormalised, so that internal divergences are eliminated, but it may develop new divergences. This issue was discussed in general terms at the end of § 4.

In this and the remaining sections we turn to specific examples. First, the function of interest in XY model is (Wiegmann 1978, José *et al* 1978) (see also appendix 5)

$$\mathscr{G}(\boldsymbol{R}) = \langle\cos\theta(\boldsymbol{R})\cos\theta(0)\rangle_{XY}$$

$$\propto \left\langle \exp\left[(2\pi/\beta_0)\int_{-\infty}^{R_1}\phi_2(z,R_2)\,dz\right]\exp\left[(2\pi/\beta_0)\int_{-\infty}^{0}\phi_2(z,0)\,dx\right]\right\rangle_{SG}$$

where $\boldsymbol{R} = (R_1, R_2)$ and $\phi_2(x, y) = \partial\phi/\partial y$. In other words, the XY correlation function is a Green function of the composite operators $\exp[(2\pi/\beta_0)\int_{-\infty}^{x_1}\phi_2(z, x_2)\,\mathrm{d}z]$. These operators have been shown to be *bona fide* boson fields (Mandelstam 1975, equation 2.5).

Since $\mathscr{G}(R)$ is invariant under rotations of $\boldsymbol{R}$, one can choose $R_2 = 0$ and $R_1 = |\boldsymbol{R}|$. $\mathscr{G}(\boldsymbol{R})$ can then be written as:

$$\mathscr{G}(\boldsymbol{R}) \propto \left\langle \exp\left[(2\pi/\beta_0)\int_0^R \phi_2(z, 0)\,\mathrm{d}z\right]\right\rangle_{\mathrm{SG}}. \tag{9.1}$$

We will focus on the XY correlations in the low-temperature XY phase, $\beta^2 > 8\pi$. Naively, one would say that since the theory is asymptotically free in this region, all we need is the free part of $\mathscr{G}(\boldsymbol{R})$. For $a \ll |\boldsymbol{R}| \ll m_0^{-1} \to \infty$ the free part is:

$$\mathscr{G}_0(\boldsymbol{R}) = [a^2(R^2 + a^2)^{-1}]^{\pi/\beta_0^2}. \tag{9.2}$$

Assuming that $\mathscr{G}(\boldsymbol{R})$ is renormalisable multiplicatively we would have (see e.g. §§ 7 and 8) $\mathscr{G}_R(\boldsymbol{R}) = Z_{XY}\mathscr{G}(\boldsymbol{R})$. In the limit $m \to 0$ the asymptotic large distance behaviour of $\mathscr{G}_R$ will therefore be given by:

$$\mathscr{G}_R(\boldsymbol{R}) = \exp\left(\int_{R^{-1}}^1 \frac{\mathrm{d}z}{z}\gamma_{XY}[\delta(z), \alpha(z)]\right)\mathscr{G}_R(R = 1, \delta(R^{-1}), \alpha(R^{-1}), \kappa) \tag{9.3}$$

where $\gamma_{XY}(\delta, \alpha) = \kappa(\partial \ln Z_{XY}/\partial\kappa)_{\mathrm{b}}$. When $R \to \infty$, the last factor in (9.4) tends to a constant, as $\alpha(R^{-1}) \to 0$, $\delta(R^{-1}) \to \delta(0)$. As was discussed in § 8, any power of δ and α in γ_{XY}, higher than 1, will not contribute to the asymptotic behaviour near the critical temperature, since both α and $\delta \to 0$ on the separatrix.

9.2. *Questions of renormalisability*

(i) Is $\mathscr{G}(R)$ renormalisable at $\beta^2 = 8\pi$?

(ii) If so, does γ_1 (of the end of § 8) get contributions linear in α or δ, beyond those implied by equation (9.2)?

The answer to (i) is positive and to (ii) is negative. We show this explicitly to $\mathrm{O}(\alpha^2)$, namely, as $m_0 \to 0$, $\mathscr{G}(R)$ has no term linear in α_0 (appendix 6). The term proportional to α_0^2 contains, in the limit $m_0 \to 0$, three types of divergences; $\ln R^2 \ln a^2$, $(\ln a^2)^2$ and $\ln a^2$. The persistence of the first one implies that $\mathscr{G}$ is not renormalisable multiplicatively. However, there is another source for such terms. It is in $\mathscr{G}_0(\boldsymbol{R})$. That all the proper cancellations take place is shown in appendix 6. *The function is renormalisable to order* α^2.

To remove the a dependence we have to choose:

$$Z_{XY} = (\kappa^2a^2)^{-1/8}[1 + (\delta/8)\ln\kappa^2a^2 - (\alpha^2/1024)\ln^2(\kappa^2a^2) + D\alpha^2\ln\kappa^2a^2] \tag{9.4}$$

where D is some constant. This leads to

$$\gamma_{XY} = \kappa(\partial \ln Z_{XY}/\partial\kappa)_{\mathrm{b}} = -\tfrac{1}{4}(1 - \delta) + \mathrm{O}(\alpha^2, \delta^2). \tag{9.5}$$

9.3. *Asymptotic behaviour—leading and next-to-leading terms*

Restricting ourselves first to the critical temperature—δ and α on the separatrix—we have, from the solution of the flow equations in lowest order (§ 8), $\delta(z) \sim$

$-(2\ln z)^{-1}$. Substituting $\delta(z)$ in (9.5) and γ_{XY} in (9.3), we find:

$$\mathscr{G}_R(\boldsymbol{R}) \sim R^{-1/4}(\ln R)^{1/8}. \tag{9.6}$$

This is just the form discovered by Kosterlitz, with the celebrated $\eta = \frac{1}{4}$. We also see that the power of $\ln R$ in $\mathscr{G}(\boldsymbol{R})$ arises as a typical remnant of asymptotic freedom. We can extend this result in two different ways:

(i) Going slightly below the critical temperature—the region to the right of the separatrix in the (α, δ) plane in figure 6. In this region the explicit solution of Kosterlitz's equations is

$$\delta(z) = \delta(0)\frac{[\delta(1)+\delta(0)]+[\delta(1)-\delta(0)]z^{4\delta(0)}}{[\delta(1)+\delta(0)]-[\delta(1)-\delta(0)]z^{4\delta(0)}} \tag{9.7}$$

where $\delta(1)$ and $\delta(0)$ are the initial and final values of δ along a given trajectory. Furthermore, $\delta(0)$ measures the departure of the temperature from the critical temperature.

Inserting (9.7) and (9.5) and in (9.3) one finds:

$$\mathscr{G}(\boldsymbol{R}) \sim R^{-\frac{1}{4}[1-\delta(0)]}|\ln R|^{1/8}. \tag{9.8}$$

In other words, η of the XY model begins to decrease as T decreases below T_c. As was mentioned in the Introduction, at very low temperature $\eta \propto T$. These two results support the hypothesis that η starts from zero at $T = 0$, increases monotonically until T_c and becomes $\frac{1}{4}$. The relation between $\delta(0)$ and $T - T_c$ is not universal, of course. Yet it clearly indicates a linear decrease in η as one penetrates the low-temperature phase.

(ii) A more exciting result is perhaps the fact that the next-to-leading terms in $\mathscr{G}(\boldsymbol{R})$ at the critical temperature are universal, and can be computed in terms of the higher order terms in the flow equations of § 7.

Returning to the critical temperature, we use the result (8.8), namely, that when $z \to 0$

$$\delta(z) \sim -\frac{1}{2\ln z} - \frac{1}{4}\frac{\ln|\ln z|}{\ln^2 z}. \tag{9.9}$$

Recall that both coefficients in (9.9) are universal. Equation (9.9) is the approximate solution, for small z, of the equation:

$$\frac{dz}{z} = \frac{d\delta}{2\delta^2(1+\frac{1}{2}\delta)} \tag{9.10}$$

and hence:

$$\int_{R^{-1}}^{1} \frac{dz}{z}\delta(z) = \frac{1}{2}\int_{\delta(R^{-1})}^{\delta(1)} \frac{d\delta}{\delta(1+\frac{1}{2}\delta)} = \frac{1}{2}\ln[\delta(1+\tfrac{1}{2}\delta)^{-1}]\Big|_{\delta(R^{-1})}^{\delta(1)}.$$

Substituting (9.9) in the last expression, and the result in (9.3) one finds:

$$\mathscr{G}(\boldsymbol{R}) \sim R^{-1/4}(\ln R)^{1/8}\left(1 + \frac{1}{16}\frac{\ln\ln R}{\ln R}\right). \tag{9.11}$$

10. Calculation and renormalisation of the free energy and the function $\langle \int \phi^2(x) \rangle$

10.1. Calculation to $O(\alpha^2)$

The bare free energy discussed at the end of § 4 is given to $O(\alpha^2)$ by

$$F_b = \tfrac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \ln(q^2 + m^2) - \alpha_0 \beta_0^{-2} J - \tfrac{1}{2}(\alpha_0 J \beta_0^{-2})^2 \int d^2 y [\cosh I(y) - 1] \qquad (10.1)$$

where J is defined in equation (3.1). The corresponding graphs are shown in figure 7.

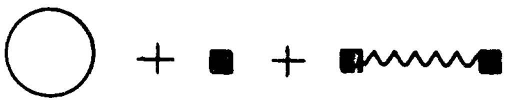

Figure 7. Graphs contributing to the free energy up to order α^2.

The first graph will, by convention, represent the first term in (10.1). From this expression it follows that

$$\begin{aligned} F'_b &= \partial F_b / \partial m_0^2 \\ &= -(8\pi)^{-1} \ln c m_0^2 a^2 - (\alpha_0 J / 8\pi m_0^2) \\ &\quad - (\beta_0^2 / 8\pi m_0^2)(\alpha_0 J / \beta_0^2)^2 \int d^2 y [\cosh I(y) - 1] \\ &\quad - \tfrac{1}{2}(\alpha_0 J / \beta_0^2)^2 \int d^2 y [\partial I(y) / \partial m_0^2] \sinh I(y). \end{aligned} \qquad (10.2)$$

This function is nothing but $\langle \int \phi^2(x) \rangle$ calculated to $O(\alpha^2)$. We proceed to renormalise it first.

Substituting $\beta_0^2 = 8\pi(1 + \delta_0)$ one finds

$$\begin{aligned} F'_b = &-(1/8\pi) \ln(c m_0^2 a^2) - (\alpha_0 c / 8\pi)[1 + \delta_0 \ln(c m_0^2 a^2)] \\ &- (c^2 \alpha^2 m_0^2 / 64\pi^2) \int d^2 y [\cosh I(y) - m_0 y K_1(m_0 y) \sinh I(y) - 1] \end{aligned} \qquad (10.3)$$

where we made use of the fact that $I \propto K_0$, and that the derivative of K_0 is $-K_1$. Further, we can set $m_0 = m$, $\alpha_0 = \alpha$ and $\delta_0 = \delta$ in terms of second order.

Using the expression

$$m y K_1(m y) = 1 + \tfrac{1}{4} m^2 y^2 (\ln c m^2 y^2 - 1) + O(m^4 y^4) \qquad (10.4)$$

the singular part of the α^2 term is found to be

$$-\frac{\alpha^2}{512\pi}[\ln(c m^2 a^2) - \tfrac{1}{2}\ln^2(c m^2 a^2)]. \qquad (10.5)$$

Note that the apparent divergence of the form a^{-2} disappeared in the difference $\cosh I(y) - \sinh I(y)$.

In § 4 we asserted that F'_b is made finite via equation (4.7). Use of (10.5) and (5.7*a*) yields

$$\begin{aligned} F'_R = &-(1/8\pi)[\ln(m^2/\kappa^2) + \alpha \delta c \ln(m^2/\kappa^2) \\ &+ (\alpha^2/128) \ln^2(m^2/\kappa^2) - (\alpha^2/64) \ln(m^2/\kappa^2)] \end{aligned} \qquad (10.6)$$

which is indeed finite. No additional renormalisation was required, in support of our contention (4.1*c*).

Let us now calculate the renormalised free energy, defined in equation (4.6). In order to perform the calculation we use the fact that

$$-\tfrac{1}{2}\int \frac{d^2q}{(2\pi)^2}\{\ln[(q^2+m_0^2)(q^2+\kappa_0^2)^{-1}]-(m_0^2-\kappa_0^2)(q^2+\kappa_0^2)^{-1}\}$$

$$=(Z_\phi^{-1}/8\pi)[m^2\ln(m^2/\kappa^2)-m^2+\kappa^2] \tag{10.7}$$

after κ_0 and m_0 are substituted in terms of κ and m. We also need $I(x)$ beyond the first leading term, namely:

$$I(x)\sim -2\ln(cm^2(x^2+a^2)]-\tfrac{1}{2}m^2(x^2+a^2)\ln[cm^2(x^2+a^2)]+m^2(x^2+a^2)+\ldots. \tag{10.8}$$

This leads to:

$$\tfrac{1}{2}[(c\alpha m^2)^2(8\pi)^{-2}]\int \cosh I(x)\,d^2x$$

$$=(\alpha^2/256\pi)a^{-2}+(\alpha^2m^2/1024\pi)\ln^2(cm^2a^2)$$

$$-(\alpha^2m^2/256\pi)\ln(cm^2a^2)+\text{finite terms}. \tag{10.9}$$

Using (10.5), one finds that the $1/a^2$ divergence is cancelled by the first subtraction and the logarithmic divergence by the second. The result is:

$$F_R=-(1/8\pi)[m^2\ln(m^2/\kappa^2)-m^2+\kappa^2+\alpha\delta cm^2\ln(m^2/\kappa^2)-\alpha\delta c(m^2-\kappa^2)$$

$$+(\alpha^2m^2/128\pi)\ln^2(m^2/\kappa^2)-(\alpha^2m^2/32)\ln(m^2/\kappa^2)]. \tag{10.10}$$

10.2. Renormalisation group equations

Since the free energy needs two subtractions (equation (4.6)) in order to be renormalised, it satisfies an inhomogeneous RGE:

$$[\kappa\partial/\partial\kappa+\beta_\alpha\partial/\partial\alpha+\beta_\delta\partial/\partial\delta+\gamma_\phi m^2\partial/\partial m^2]F_R(\alpha,\delta,m^2,\kappa)$$

$$=-\kappa^2(m^2-\kappa^2)[2-\gamma_\phi(\alpha,\delta)]F_R''(\alpha,\delta,\kappa^2,\kappa) \tag{10.11}$$

where F'' denotes a double derivative with respect to m^2.

The solution of this equation is given by

$$F_R(\alpha,\delta,m^2,\kappa)=-\kappa^2\int_\rho^1 x\,dx\,[m^2(x)-\kappa^2][2-\gamma_\phi(\alpha(x),\delta(x))]F_R''[\alpha(x),\delta(x),\kappa^2,\kappa]$$

$$+\rho^2F_R[\alpha(\rho),\delta(\rho),m^2(\rho),\kappa]. \tag{10.12}$$

The function $m^2(x)$ is given in equation (8.11). For $m=0$ this reduces to

$$F_R(\alpha,\delta,0,\kappa)=\kappa^4\int_\rho^1 x\,dx\,[2-\gamma_\phi(x)]F_R''[\alpha(x),\delta(x),\kappa^2,\kappa]+\rho^2F_R[\alpha(\rho),\delta(\rho),0,\kappa]. \tag{10.13}$$

On the other hand, we know from perturbation theory that

$$F_R''(\alpha,\delta,\kappa^2,\kappa)=(1/8\pi\kappa^2)[1+O(\alpha^2,\alpha\delta)]. \tag{10.14}$$

The imhomogeneous term will therefore behave as $-(\kappa^2\rho^2)/8\pi$. Choosing ρ by $\delta(\rho)=1$ one finds (see § 7) that

$$\rho=\xi^{-1}.$$

Assuming that $F_R(1, 1, 0, \kappa)$ is non-singular, it follows that the leading singularity of the free energy is given by $F\propto\xi^{-2}$, in agreement with Kosterlitz (1974). The second derivative of F with respect to m^2 which is just $\langle\int\phi^2(x)\int\phi^2(y)\rangle$ satisfies the RGE

$$(\kappa\partial/\partial\kappa+\beta_\alpha\partial/\partial\alpha+\beta_\delta\partial/\partial\delta-2\gamma_\phi)F''(\alpha, \delta, 0, \kappa)=0$$

from which it follows that the scale dimension of ϕ^2 on the line of fixed points $\alpha=0$ is zero.

11. Harmonic perturbations

In the present section we study the question of the effect of adding a perturbation of the type $\cos n\beta_0\phi$ to the Lagrangian. Our considerations so far indicate that such terms will not be generated by the renormalisation process, since we have found that two renormalisation constants suffice. Such perturbations are, however, present if one considers (Kosterlitz 1974) vortices of vorticity higher than unity (Samuel 1978).

We now show that all these operators, with $n>1$, are irrelevant near $\beta^2=8\pi$. In other words, their scale dimension will turn out to be greater than 2—the dimension of $\cos\beta_0\phi$.

Consider the function

$$C_n(\boldsymbol{R})=\langle\exp\{i\beta_0 n[\phi(\boldsymbol{R})-\phi(0)]\}\rangle_{SG}. \tag{11.1}$$

In region I of figure 6 the dimension of C_n will be determined by the free theory, provided the function in renormalisable.

The free function is given by

$$C_n^0(\boldsymbol{R})=\exp\{n^2[I(R)-I(0)]\}\sim(a^2/R^2)^{n^2\beta_0^2/4\pi}. \tag{11.2}$$

The proof that C_n is renormalisable to $O(\alpha^2)$ is virtually identical to that for the XY correlation function of § 9 and appendix 6. Apart from the subtraction of power divergences, which originate from the coupling of $\exp(i\beta_0 n\phi)$ to similar operators with lower n, shifting the various parameters. This is analogous to the renormalisation of high composite operators in the ϕ^4 theory (Brezin *et al* 1976, § III.4.b).

Defining Z_n^2 to be the renormalisation constant for C_n, we find (appendix 7)

$$\gamma_n=-\kappa\partial\ln Z_n/\partial\kappa=2n^2(1+\delta)+O(\alpha^2). \tag{11.3}$$

The scale dimension of $\exp(in\beta\phi)$ will therefore be $2n^2$ at the critical point $\alpha=\delta=0$. This will be the dimension of $\cos n\beta\phi$ as well. Clearly, for $n>1$, this operator is irrelevant. It will not affect the critical behaviour of the XY model, since starting on any point on the critical line (separatrix) in the α, δ plane one is driven to the origin $\alpha=\delta=0$.

This establishes our statement that vortices with higher vorticity than 1, which should in principle be present when the XY model is transformed into a field theory (see Introduction), are irrelevant.

12. Symmetry breaking fields

We now consider in contrast to the operators $\cos n\beta\phi$, where ϕ is the SG field, operators of the form $\cos p\theta(x)$ (José *et al* 1978), where $\theta(x)$ is the XY angle variable and p is an integer. Such operators are important since crystal fields always break the planar rotational symmetry.

To study these operators we consider again $\exp[ip\theta(x)]$. In appendix 5 it was shown that the XY operator $\exp[i\theta(x)]$ is translated into the SG operator $\exp[(2\pi/\beta_0)\int_{-\infty}^{x_1}(\partial\phi/\partial z_2)\,dz_1|_{z_2=x_2}]$. Thus the symmetry breaking operator will translate via:

$$0_p = \exp[ip\theta(x)] \rightarrow \exp\left[(2\pi p/\beta_0)\int_{\infty}^{x_1}(\partial\phi/\partial z_2)\,dz_1\Big|_{z_2=x_2}\right]. \qquad (12.1)$$

The analysis of § 9 and appendix 6 can now be repeated, with β_0 replaced by β_0/p. Hence the correlation function $\langle 0_p(x)0_p(y)\rangle_{SG}$ will renormalise multiplicatively, with renormalisation constant Z_p^2. The γ function associated with it is:

$$\gamma_p = -\kappa\,\partial \ln Z_p/\partial\kappa = (p^2/8)(1-\delta)+O(\alpha^2). \qquad (12.2)$$

It therefore follows that the scale dimension of 0_p, at the critical temperature, is

$$d_p = p^2/8. \qquad (12.3)$$

0_p is relevant only if $p^2/8<2$ or $p<4$ and is marginal for $p=4$, as was recognised by José *et al* (1978).

Using the logic of § 7.3 the flow equation for h_p, the renormalised coupling constant associated with the operator 0_p in the Lagrangian, follows directly from (12.2) and the definition, $h_p^0 = Z_p h_p$, of h_p in terms of the bare coupling constant ('symmetry breaking field') h_p^0

$$\rho\,d\ln h_p(\rho)/d\rho = -2+\gamma_p[\alpha(\rho), \delta(\rho)] \qquad (12.4)$$

with $h_p(1)=h_p$.

This equation has the solution

$$h_p(\rho) = \rho^{-2}h_p \exp\left(\int_\rho^1 \frac{dx}{x}\gamma_p[\alpha(x), \delta(x)]\right). \qquad (12.5)$$

Note that a flow equation of the type (12.6) was derived in José *et al* (1978). However, as on previous occasions (Amit and Goldschmidt 1978, Amit *et al* 1978), the field theoretical technique allows a choice of renormalisation which decouples this equation from the rest of the flow equations for α and δ.

Finally, using our results for γ_p we can obtain directly the critical exponent $\delta=15$ (Kosterlitz 1974).

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Appendix 1

We calculate here the divergent part of $\Gamma_b^{(2)}$ which is proportional to α^2 and $\alpha^2\delta$. This divergent term has no p^2-independent part. When $\exp(ipx)$ (see equation (3.*b*)) is expanded in powers of p, the odd terms in p vanish when integrated over angles.

The coefficient of p^2 is divergent. We have to compute

$$(\alpha_0^2 J^2/4\beta_0^2)\int_0^{\infty} \mathrm{d}^2x\,(px)^2 \exp\{I(x)\}$$

$$= (1/32)p^2\alpha_0^2(cm_0^2)^2[1-\delta_0+2\delta_0\ln(cm_0^2a^2)]\int_0^{m_0^{-1}} \frac{x^3\,\mathrm{d}x}{[cm_0^2(x^2+a^2)]^{2+2\delta_0}}$$
$$+\text{finite terms.} \qquad \text{(A1.1)}$$

The integral is calculated as follows:

$$2(cm_0^2)^2\int_0^{m_0^{-2}} \frac{x^3\,\mathrm{d}x}{[cm^2(x^2+a^2)]^{2+2\delta_0}} = \int_0^{m_0^{-2}} \frac{z\,\mathrm{d}z}{(z+a^2)^2}[1-2\delta_0\ln cm_0^2(z+a^2)]$$
$$= -\ln(cm_0^2a^2)+\delta_0\ln^2(cm_0^2a^2)+2\delta_0\ln(cm_0^2a^2)+\text{constant.} \qquad \text{(A1.2)}$$

Substituting (A1.2) in (A1.1) we reach the result (5.3).

Appendix 2

In this appendix we prove that there is no divergent contribution of $O(\alpha^3)$ to the p^2 term in $\Gamma^{(2)}$. We have to consider the expression

$$[(\alpha_0 cm_0^2)^3(8\pi)^{-2}]p^2\int \cos^2\theta_x x^2\,\mathrm{d}^2x\,\mathrm{d}^2y\{\sinh I(x)[I(y)-\tfrac{1}{2}]$$
$$+\tfrac{1}{2}[\sinh I(x)\cosh I(y)\cosh I(x-y)-\cosh I(x)\sinh I(y)\sinh I(x-y)]\}. \qquad \text{(A2.1)}$$

As mentioned in § 5, divergences can only arise from the four regions of integration:

(1) $\{|x|, |y|, |x-y|<\Delta\}$; (2) $\{|x|<\Delta; |y|, |x-y|>\Delta\}$;

(3) $\{|y|<\Delta; |x|, |x-y|>\Delta\}$; (4) $\{|x-y|<\Delta; |x|, |y|>\Delta\}$.

Since $m^2\Delta^2 \ll 1$, in the region $|x|<\Delta$ one can use the approximation

$$I(x) \sim -2\ln[cm_0^2(x^2+a^2)] \qquad \text{(A2.2)}$$

after having set $\beta_0^2 = 8\pi$.

Since the overall divergences must be Δ-independent we consider only the Δ-independent divergent terms. The calculation over the inner region $\{|x|, |y|, |x-y|<\Delta\}$ is rather straightforward, since all propagators can be approximated by (A2.2). The result is proportional to Δ and can be ignored.

To integrate over the remaining regions, first note that the change of variable $\boldsymbol{y} \leftrightarrow \boldsymbol{y}-\boldsymbol{x}$ makes the divergent contributions of regions (3) and (4) identical. In regions (2) and (3) *either* (x) or (y) is always $<\Delta$, so that (A2.2) is applicable for that variable. The other variable can be integrated over the range $(0, \infty)$, rather than the prescribed (Δ, ∞). The difference is of $O(\Delta)$.

In the region $|x|<\Delta$:

$$I(x+y)=I(y)+hI'(y)+\tfrac{1}{2}h^2I''(y)+\dots \tag{A2.3}$$

where $I'(y)=(\mathrm{d}/\mathrm{d}y^2)I(y)$ and $h=2xy+x^2$. A similar expansion, in powers of $k=2\boldsymbol{x}-\boldsymbol{y}+y^2$, will hold in powers of k when $|y|<\Delta$.

Using this expansion one can expand the hyperbolic 'propagators' as

$$\sinh I(x+y)=\cosh I(y)[hI'(y)+\tfrac{1}{2}h^2I''(y)]+\sinh I(y)[1+\tfrac{1}{2}h^2(I'(y))^2]+O(h^3)$$

$$\cosh I(x+y)=\cosh I(y)[1+\tfrac{1}{2}h^2I'(y)]^2+\sinh I(y)[hI'(y)+\tfrac{1}{2}hI''(y)]+O(h^3). \tag{A2.4}$$

Similar expressions hold for the expansion in powers of k.

We denote the divergent contributions from regions (2) and (3) by J_2 and J_3, respectively, apart from the coefficient $[(\alpha_0 cm_0^2)^3(8\pi)^{-2}]p^2$.

$$J_2=\int \mathrm{d}^2x\,\cos^2\theta_x x^2\,\mathrm{d}^2y\,\frac{1}{2(cm^2)^2(x^2+a^2)^2}\{I(y)-\tfrac{1}{2}+\tfrac{1}{2}[\cosh^2 I(y)-\sinh^2 I(y)]+O(x^2)\}. \tag{A2.5}$$

The $O(x^2)$ terms are non-singular and can be omitted. Therefore, substituting (2.2) for I, we are left with

$$J_2\sim\frac{4\pi^2}{(cm^2)^2}\int_0^\Delta \frac{x^3\,\mathrm{d}x}{(x^2+a^2)^2}\int_0^\infty y\,\mathrm{d}y\,K_0(my)=-\frac{2\pi^2c}{(cm^2)^3}\ln(a^2/\Delta^2) \tag{A2.6}$$

where we have set $a=0$ in K_0, and used Gradshteyn and Ryzhik (1965, equation 6.561.16).

In region (3) only the expression in the last square brackets in equation (A2.1) gives a divergent contribution. Using the expansion in k, discussed above, it is easy to bring the divergent contribution into the form:

$$J_3=-\frac{2\pi}{(cm^2)^2}\int \mathrm{d}^2x\,\cos^2\theta_x x^2\,\mathrm{d}^2y\,\frac{1}{(y^2+a^2)^2}[y^2I'(x)+2x^2y^2\cos^2\theta_y I''(x)]. \tag{A2.7}$$

Using the relations

$$I'(x)=\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}K_0(mx) \tag{A2.8}$$

$$I''(x)=\frac{1}{x^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}K_0(mx)-\frac{1}{x^3}\frac{\mathrm{d}}{\mathrm{d}x}K_0(mx) \tag{A2.9}$$

and the differential equation satisfied by K_0, the integrals in (A2.7) are evaluated (Gradshteyn and Ryzhik 1965) to give

$$J_3 = \frac{\pi^2 c}{(cm^2)^3} \ln(a^2/\Delta^2). \tag{A2.10}$$

The overall divergent contribution, $J_2 + 2J_3$, is therefore zero.

Appendix 3

The divergent contribution of $O(\alpha^3)$ to the p-independent part of $\Gamma^{(2)}$ is calculated by evaluating the expression

$$\frac{1}{512\pi^2}\alpha_0^3(cm_0^2)^3 \int d^2x\, d^2y\, \{\exp[I(x)+I(y)-I(x+y)] - \exp I(x) - \exp I(y)\}. \tag{A3.1}$$

The two regions $\{|x|<\Delta, |y|>\Delta\}$ and $\{|y|<\Delta, |x|>\Delta\}$ (regions 1 and 2 in (5.4a)) give equal divergent contributions. In the first region we use the expansion (A2.3) for $I(x+y)$. The part of the integrand in (A3.1) which contributes to the divergence is

$$\exp I(x)\{\exp[-hI'(y) - \tfrac{1}{2}h^2 I''(y)] - 1\}. \tag{A3.2}$$

Expanding the y-dependent exponentials up to $O(x^2)$, (A3.2) becomes:

$$\exp[I(x)][-hI'(y) + \tfrac{1}{2}h^2[I'(y)]^2 - \tfrac{1}{2}h^2 I''(y)]. \tag{A3.3}$$

Using relations (A2.2), (A2.8) and (A2.9), the differential equation satisfied by K_0, as well as the relation $dK_0/dx = -K_1$, integrating over the angles, the contribution of this region to (A3.1) becomes

$$J_1 = \frac{\alpha_0^3 cm_0^2}{128} \int x^3\, dx \frac{1}{(x^2+a^2)^2} \int_\Delta^\infty y\, dy [4m_0^2 K^2(m_0^2 y) - m^2 K_0(m_0 y)].$$

Using Gradshteyn and Ryzhik (1965, equations 5.52.1 and 5.54.2), one finds

$$J_1 = -\frac{\alpha_0^2 cm_0^2}{256} \ln(a^2/\Delta^2)\{2y^2[K_1^2(y) - K_0(y)K_1(y)] + yK_1(y)\}|_{m_0\Delta}^\infty$$

where the fact that $\Delta \gg a$ permits us to take $a=0$ in the K. Finally, the divergent contribution of this region is

$$J_1 = \frac{\alpha_0^3 cm_0^2}{128} \ln(a^2/\Delta^2)[\ln(cm_0^2\Delta^2) + \tfrac{3}{2}]. \tag{A3.4}$$

On adding the identical contribution of region (2), we obtain

$$J_1 + J_2 \approx \frac{\alpha_0^2 cm_0^2}{64} \ln(a^2/\Delta^2)[\ln(cm_0^2\Delta^2) + \tfrac{3}{2}] \tag{A3.5}$$

In region (3) $\{|x|, |y|<\Delta\}$, (A3.1) can be approximated by

$$\begin{aligned} J_3 &\approx \frac{\alpha_0^3 cm_0^2}{512\pi^2} \int d^2x\, d^2y \left[\left(\frac{(x-y)^2+a^2}{(x^2+a^2)(y^2+a^2)}\right)^2 - \frac{2}{(x^2+a^2)^2}\right] \\ &= \frac{\alpha_0^3 cm_0^2}{128} \ln^2(a^2/\Delta^2) + \frac{\alpha_0^3 cm_0^2}{64} \ln(a^2/\Delta^2) + \text{finite terms}. \end{aligned} \tag{A3.6}$$

Adding (A3.5) and (A3.6) one obtains for the divergent contribution

$$\frac{\alpha_0^3 cm_0^2}{128}\ln^2 a^2+\frac{\alpha_0^3 cm_0^2}{64}\ln a^2 \ln(cm_0^2)+5\frac{\alpha_0^3 cm_0^2}{128}\ln a^2. \tag{A3.7}$$

Note the important result that *the* $\ln a^2 \ln \Delta^2$ *term has disappeared*! Since the final result cannot depend on Δ, the total divergent contribution to (A3.1) must be of the form

$$J=(\alpha_0^3 cm_0^2/128)[\ln^2(cm_0^2 a^2)+5\ln(cm_0^2 a^2)]. \tag{A3.8}$$

This is equation (5.5).

Appendix 4

The bare two-point function is given by

$$\begin{aligned}\Gamma_B^{(2)}=p^2+m_0^2&+\alpha_0 cm_0^2[1+\delta_0\ln(cm_0^2a^2)+\tfrac{1}{2}\delta_0^2\ln^2(cm_0^2a^2)]\\&+c_1\alpha_0^2cm_0^2[1+2\delta_0\ln(cm_0^2a^2)]\\&-(1/64)\alpha_0^2p^2(1-\delta_0)[1+2\delta_0\ln(cm_0^2a^2)][\ln(cm_0^2a^2)-\delta_0\ln^2(cm_0^2a^2)\\&-2\delta_0\ln(cm_0^2a^2)+c_2+c_3\delta_0]\\&+(1/128)\alpha^3cm_0^2[\ln^2(cm_0^2a^2)+5\ln(cm_0^2a^2)+c_4]\end{aligned} \tag{A4.1}$$

where c_i are some finite constants.

Using the definitions (4.1) and (4.2), of Z_α and Z_ϕ, we obtain the following expression for the renormalised two-point function:

$$\begin{aligned}\Gamma_R^{(2)}=Z_\phi p^2+m^2&+\alpha cm^2Z_\alpha[1+\delta\ln(cm^2a^2)+(Z_\alpha^{-1}-1)\ln(cm^2a^2)+\tfrac{1}{2}\delta^2\ln^2(cm^2a^2)\\&+c_1\alpha^2cm^2Z_\alpha^2[1+2\delta\ln(cm^2a^2)]\\&-(1/64)\alpha^2p^2Z_\alpha^2(1-\delta)[1+2\delta\ln(cm^2a^2)][\ln(cm^2a^2)\\&-\delta\ln^2(cm^2a^2)-2\delta\ln(cm^2a^2)+c_2+c_3\delta]\\&+(1/128)\alpha^3cm^2[\ln^2(cm^2a^2)+5\ln(cm^2a^2)+c_4]+O(\alpha^4).\end{aligned} \tag{A4.2}$$

The expansions of the renormalisation constants in terms of the renormalised α and δ,

$$Z_\phi=1+B_1\alpha^2+B_2\alpha^2\delta+B_3\alpha^3 \tag{A4.3}$$

$$Z_\alpha=1+A_1\delta+A_2\alpha^2+A_3\delta^3 \tag{A4.4}$$

are substituted in (A4.2). It is then rearranged in such a way that the coefficients of p^2 and m^2 are ordered in the double expansion in α and δ.

We find

$$\begin{aligned}\Gamma_R^{(2)}=p^2+m^2&+\alpha cm^2+\alpha\delta cm^2[\ln(cma^2)+A_1]+c_1\alpha^2cm^2-(1/64)\alpha^2p^2[\ln(cm^2a^2)\\&-64B_1]+c_1(cm^2)\alpha^2\delta[\ln(cm^2a^2)+A_1]+B_3\alpha^3p^2\\&-(1/64)\alpha^2\delta p^2[2A_1\ln(cm^2a^2)-3\ln(cm^2a^2)+\ln^2(cm^2a^2)\\&+2c_2[A_1+\ln(cm^2a^2)]-c_2+c_3-64B_2\}+\tfrac{1}{2}\alpha\delta^2cm^2[\ln^2(cm^2a^2)\end{aligned}$$

$$+2A_1 \ln(cm^2a^2)+2A_3]+(1/128)\alpha^3m^2c[128A_2-128B_1\ln(cm^2a^2)$$
$$+\ln^2(cm^2a^2)+5\ln(cm^2a^2)+c_4]. \quad \text{(A4.5)}$$

This function is made finite by choosing

$$A_1=-\ln(\kappa^2a^2) \qquad B_1=(1/64)\ln(\kappa^2a^2)$$
$$A_2=(1/128)\ln^2(\kappa^2a^2)-(5/128)\ln(\kappa^2a^2) \qquad B_2=(1/64)\ln^2(\kappa^2a^2)-(3/64)\ln(\kappa^2a^2)$$
$$A_3=\tfrac{1}{2}\ln^2(\kappa^2a^2) \qquad B_3=0.$$

These are just the results discussed in § 5.

Note the following features which appear in the process. As was mentioned in § 5, the determination of Z_α or of A_1 eliminates three divergences. Moreover, the cancellation of subdivergences by insertions in lower order terms, manifests itself in three instances: (a) the determination of A_1 eliminates the $\ln^2$ divergences in the term proportional to $\alpha^2\delta^2p^2$, and (b) in $\alpha\delta^2m^2$, (c) the determination of B_1 cures the $\ln^2$ in the last term in (A4.5).

Appendix 5

Equation (9.1) expressed the XY correlation function in terms of averages over composite operators of the SG theory. Since R is taken fixed and $m_0\to 0$, all functions which will appear in the expansion of $\mathcal{G}(R)$ can be approximated by their leading behaviour for m_0R small.

We shall verify (9.1a) in perturbation theory, using a two-dimensional volume of linear size L and making $m_0\to 0$. In fact, as is shown in appendix 6, this interchange of limits can be affected in $\mathcal{G}(R)$, even in perturbation theory (see e.g. Elitzur 1979).

The expression for $\mathcal{G}(R)$ in perturbation theory is:

$$\mathcal{G}(\boldsymbol{R})=[Z^{-1}]\sum_{n=0}^{\infty}\frac{1}{(n!)^2}(\alpha_0/2\beta_0^2a^2)^{2n}\int d^2x^{(1)}\dots d^2x^{(2n)}\int D\phi$$
$$\times\exp\left[(2\pi/\beta_0)\int_0^R\phi_2(z,0)\,dz\right]\exp\left[i\beta_0\sum_{i=1}^{2n}S_i\phi(x^{(i)})\right]$$
$$\times\exp\left[-\tfrac{1}{2}\int d^2x\,d^2y\phi(\boldsymbol{x})G_0^{-1}(\boldsymbol{x}-y)\phi(\boldsymbol{y})\right] \quad \text{(A5.1)}$$

where Z is the partition function of the SG theory. In (A5.1) we have already taken into account the well-known fact that only even powers of α survive as $m_0\to 0$ (Coleman 1975). In fact, from the powers of $\cos\phi$, entering the expansion, only products of exponentials with equal numbers of $i\phi$ and $(-i\phi)$, contribute Counting these 'neutral' combinations transforms the $(2n!)^{-1}$ into $(n!)^{-2}$.

The S_i express the 'neutrality'. They are simply defined as $S_i=1(i=1,\dots,n)$; $S_i=-1(i=n+1,\dots,2n)$.

The first and second exponentials, inside the functional integral in (A5.1), are exponentials of linear operators acting on ϕ. Hence their product can be written as

$\exp\int d^2z\, J_S(z)\phi(z)$, with

$$J_S(\boldsymbol{z}) = -(2\pi/\beta_0)\delta'(z_2)\bar{\theta}[z_1, (0, R)] + i\beta_0 \sum_{i=1}^{2n} S_i\delta(\boldsymbol{z} - \boldsymbol{x}^{(i)}). \tag{A5.2}$$

Here we have used the following notation: δ' is a derivative of a one-dimensional δ function. The function $\bar{\theta}$ is defined as:

$$\bar{\theta}[z, (a, b)] = \begin{cases} 1 & a \leqslant z \leqslant b \\ 0 & \text{otherwise.} \end{cases} \tag{A5.3}$$

The Gaussian integrals over the ϕ in (A5.1) can be performed. The result is:

$$\mathscr{G}(\boldsymbol{R}) = (Z_0/Z) \sum_{n=0}^{\infty} (n!)^{-2}(\alpha_0/2\beta_0^2a^2)^{2n} \int_0^L \prod_{i=1}^{2n} d^2x^{(i)} \times \exp[\tfrac{1}{2}\int d^2x\, d^2y\, J_S(x)G_0(x-y)J_S(y)]. \tag{A5.4}$$

Z_0 is the partition function of the free SG theory.

In the limit $m_0 \to 0$, G_0 is approximated by (2.8), and

$$\int [\partial G_0(\boldsymbol{z})/\partial z_2]\, dz_1 \sim -(1/2\pi)\frac{z_2}{(z_2^2+a^2)^{1/2}} \tan^{-1}[z_1/(z_2^2+a^2)^{1/2}] \tag{A5.5}$$

$$\int [\partial^2 G_0(\boldsymbol{z}-\boldsymbol{z}')/\partial z_2\partial z_2']\, dz_1\, dz_1' \sim (1/4\pi) \ln(z^2+a^2). \tag{A5.6}$$

The exponential in (A5.4) becomes:

$$\exp\Big\{-(\pi/\beta_0^2)\ln[(R^2+a^2)/a^2] - i\sum_{i=1}^{2n} S_i[\tan^{-1}(R-x_1^{(i)})/x_2^{(i)} + \tan^{-1} x_1^{(i)}/x_2^{(i)}] + (\beta_0^2/8\pi)\sum_{i,j} S_iS_j \ln[(x^{(i)}-x^{(j)})^2+a^2]\Big\}. \tag{A5.7}$$

Substituting in (A5.4) one has:

$$\mathscr{G}(R) = [(R^2+a^2)/a^2]^{-\pi/\beta_0^2}(Z_0/Z)\sum_0^{\infty}(n!)^{-2}(\alpha_0/2\beta_0^2)^{2n} \times \int_0^L \Pi \frac{d^2x^{(i)}}{a^2} \exp i[\theta(\boldsymbol{R}) - \theta(0)] \exp(\beta_0^2/8\pi) \times \sum_{i,j} S_iS_j \ln[(x^{(i)}-x^{(j)})^2+a^2] \tag{A5.8}$$

where $\theta(\boldsymbol{R})$ is (Chui and Lee 1975)

$$\theta(\boldsymbol{R}) = \sum_{i=1}^{2n} S_i \tan^{-1}[(R_1 - x_1^{(i)})(R_2 - x_2^{(i)})^{-1}]. \tag{A5.9}$$

To calculate Z_0/Z one simply sets $R = 0$ on the RHS of (A5.8) and I on the LHS. The result is:

$$Z/Z_0 = \sum_{n=0}^{\infty} (\alpha_0/2\beta_0^2)^{2n}(n!)^{-2} \int \Pi \frac{d^2x^{(i)}}{a^2} \exp\Big\{(\beta_0^2/8\pi)\sum_{ij} S_iS_j \ln[(\boldsymbol{x}^{(i)}-\boldsymbol{x}^{(j)})^2+a^2]\Big\}. \tag{A5.10}$$

From (A5.8) and (A5.10) one can read off that:

$$\mathscr{G}(\boldsymbol{R}) = [(R^2+a^2)/a^2]^{-\pi/\beta_0^2} \langle \exp \mathrm{i}[\theta(R)-\theta(0)]\rangle_{\mathrm{CG}} \tag{A5.11}$$

where the subscript CG indicates a Coulomb gas grand ensemble of systems with equal numbers of positive and negative charges of equal magnitude, interacting via a softened logarithmic interaction, at a temperature and fugacity given by (1.2).

The first factor in (A5.11) is the spin-wave contribution. The second part represents the vortices. Thus (Kosterlitz 1974, José *et al* 1978), (A5.1) is indeed the XY correlation function in the SWCG approximation.

From (A5.8) and (A5.10) it follows that the limit $L \to \infty$ can be taken in every order in α_0. In other words, the integrals and the subintegrals converge at large $x^{(i)}$. This is in contrast to the behaviour of the SG Green function, $\Gamma^{(2)}$, which had to be resumed before the limit $m \to 0$ can be taken. Perturbation theory for $\Gamma^{(2)}$, at any fixed order, has infrared divergences. An inspection of a few higher orders in the above procedure suggests that the limit $m \to 0$ can be taken in every order in the perturbation theory for $\mathscr{G}(R)$. The situation is quite reminiscent of the σ model (see e.g. Brezin *et al* 1976, Elitzur 1979).

Using similar techniques it is possible to show that the limits $m_0 \to 0$ and $L \to \infty$ can be interchanged in every order of the perturbation expansion for $\mathscr{G}(R)$.

Appendix 6

In this appendix we study $\mathscr{G}(\boldsymbol{R})$ of equation (9.1) to verify its renormalisability to $\mathrm{O}(\alpha^2)$ and to derive γ_{XY} of § 9 to $\mathrm{O}(\alpha, \delta)$. We emphasise that since (appendix 5) $\mathscr{G}(\boldsymbol{R})$ is infrared finite in each order of perturbation theory even with $L = \infty$ and $m_0 = 0$, there is no need to introduce any IR cut-off.

Equations (A5.9) and (A5.11) imply that to $\mathrm{O}(\alpha_0^2)$:

$$\mathscr{G}(\boldsymbol{R}) = \mathscr{G}_0(\boldsymbol{R})\bigg[1 + (\alpha_0/2\beta_0^2 a^2)^2 \int_{-\infty}^{\infty} \mathrm{d}^2x^{(1)}\, \mathrm{d}^2x^{(2)} [f(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{R} - 1]$$
$$\times \{a^2[(\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)})^2 + a^2]^{-1}\}^{\beta_0^2/4\pi}\bigg] \tag{A6.1}$$

where $\mathscr{G}_o(\boldsymbol{R}) \equiv [(R^2+a^2)/a^2]^{-\pi/\beta_0^2}$,

$$f(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{R})$$
$$\equiv \frac{[R_1 - x_1^{(1)} - \mathrm{i}(R_2 - x^{(1)})][R_1 - x_1^{(2)} + \mathrm{i}(R_2 - x^{(2)})](x_1^{(1)} + \mathrm{i}x_2^{(1)})(-x_1^{(2)} + \mathrm{i}x_2^{(2)})}{[R_1 - x_1^{(1)} + \mathrm{i}(R_2 - x_2^{(1)})][R_1 - x^{(2)} - \mathrm{i}(R_2 - x_2^{(2)})](x_1^{(1)} - \mathrm{i}x_2^{(1)})(-x_1^{(2)} - \mathrm{i}x_2^{(2)})} \tag{A6.2}$$

To $\mathrm{O}(\alpha^2)$ we can set $\beta_0^2 = \beta^2 = 8\pi$, and $\alpha_0 = \alpha$ in the second-order term in (A6.1). When $\beta_0^2 = 8\pi$ the integral converges in the IR, namely, for large values of $x^{(1)}$ and $x^{(2)}$, both the double integral, as well as its subintegrals, converge. The subintegrals converge due to the last factor in the integrand, which behaves like $|x|^{-4}$. The double integral is helped by the fact that as $|x_1|, |x_2| \to \infty$, $(f-1) \to 0$. This is probably a general feature, as was discussed at the end of appendix 5.

To verify the renormalisability of $\mathscr{G}(\boldsymbol{R})$ to $\mathrm{O}(\alpha^2)$ we must check that $\mathscr{G}(\boldsymbol{R})^{1/2}$ has no $\boldsymbol{R}$-dependent divergences. Such divergences come from two sources: (a) from $\mathscr{G}_0(\boldsymbol{R})$, which is due to the renormalisation of β_0, and to $O(\alpha^2)$ is given by (via (4.1b) and (5.7a))

$$\mathscr{G}_0(\boldsymbol{R}) = [(R^2+a^2)/a^2]^{-\pi/\beta^2}[1-(\alpha^2/128)\ln(\kappa a)\ln(R/a)] \qquad \text{(A6.3)}$$

and (b) from the integral in (A6.1).

In order to analyse the integral we make the change of variable $\boldsymbol{x}^{(1)} \to \boldsymbol{w} \equiv \boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}$. It is then clear that divergent terms arise only from the three regions:

(1) $\{|\boldsymbol{w}|<\Delta, |\boldsymbol{x}^{(2)}|, |\boldsymbol{R}-\boldsymbol{x}^{(2)}|>\Delta\}$; (2) $\{|\boldsymbol{w}|<\Delta; |x^{(2)}|<\Delta\}$;

(3) $\{|\boldsymbol{w}|<\Delta; |\boldsymbol{R}-\boldsymbol{x}^{(2)}|<\Delta\}$.

(It is assumed that $|\boldsymbol{R}| \gg \Delta$.)

Regions (2) and (3) are readily seen to contribute no $\boldsymbol{R}$-dependent divergences (i.e. no $\ln R \ln a$ terms). In region (1), $f(\boldsymbol{w}, \boldsymbol{x}^{(2)}, \boldsymbol{R})$ can be expanded in powers of $|\boldsymbol{w}|/|\boldsymbol{x}|^{(2)}$ and $|\boldsymbol{w}|/|\boldsymbol{R}-\boldsymbol{x}^{(2)}|$ so that the second term in the square of (A6.1) becomes

$$-(\alpha R^2/32\pi)^2 \int_{x>\Delta} d^2x \frac{1}{(x^2+a^2)[(R-x)^2+a^2]} \int_{|w|<\Delta} d^2w \frac{1}{(w^2+a^2)}$$
$$\sim (\alpha^2/128)\ln(R/\Delta)\ln(a/\Delta). \qquad \text{(A6.4)}$$

The divergent contributions of regions (2) and (3) are identical since they are related by the variable change $\boldsymbol{x}^{(2)} \leftrightarrow \boldsymbol{R}-\boldsymbol{x}^{(2)}$. Both $\ln^2$ and $\ln a$ terms occur. The former are easy to extract, since the integration region is infinitesimal. The latter (as we shall see) are irrelevant for our purposes. Regions (2) and (3) then contribute a total divergence of $\alpha^2[-\ln^2(a/\Delta)/256 + D\ln(a/\Delta)]$ for some constant D. The overall divergent part of $\mathscr{G}(\boldsymbol{R})$ can then be written:

$$\mathscr{G}(\boldsymbol{R}) \sim (R/a)^{-2\pi/\beta^2}[1+(\alpha^2/256)\ln^2(\kappa a)+D\alpha^2\ln a]. \qquad \text{(A6.5)}$$

Note that all R-dependent divergences have cancelled. Furthermore, no IR problem is present. $\mathscr{G}(\boldsymbol{R})$ is clearly renormalisable by the multiplicative factor:

$$Z_{XY} = (\kappa a)^{-2\pi/\beta^2}\{1-[(\alpha^2/256)\ln^2(\kappa a)+D\alpha^2\ln\kappa a]\} \qquad \text{(A6.6)}$$

Since $\beta^2/8\pi = 1+\delta$ one finds:

$$\gamma_{XY} \equiv \kappa(\partial \ln Z_{XY}/\partial\kappa) = -\tfrac{1}{4}(1-\delta)+O(\alpha^2, \delta^2) \qquad \text{(A6.7)}$$

as claimed in § 9.

Appendix 7

We consider the function

$$C_n(\boldsymbol{R}) = \langle \exp\{in\beta_0[\phi(\boldsymbol{R})-\phi(0)]\}\rangle_{SG}. \qquad \text{(A7.1)}$$

Using the techniques described in appendices 5 and 6 one finds first that

$$C_n^0(R) = \exp\left\{-\tfrac{1}{2}n^2\beta_0^2 \int d^2x\, d^2y[\delta(x-R)-\delta(x)]G_0(x-y)[\delta(y-R)-\delta(y)]\right\}. \qquad \text{(A7.2)}$$

Equation (A7.2) leads directly to (11.2).

Next we proceed to compute the overlapping divergence in the second-order term; the first-order term vanishes in the limit $m_0 \to 0$. We write

$$C_n(R) = C_n^0 + \tfrac{1}{2}\alpha_0^2\beta_0^{-4}a^{-4}C_n^{(2)}(R) + \dots \tag{A7.3}$$

with

$$C_n^{(2)}(R) = \int d^2x\, d^2y \left\{ \int \langle \exp in\beta_0[\phi(\boldsymbol{R}) - \phi(0)] \cos\beta_0\phi(x) \cos\beta_0\phi(y)\rangle_0 \right.$$
$$\left. - C_n^0(\boldsymbol{R})\langle \cos\beta_0\phi(x) \cos\beta_0(y)\rangle_0 \right\}. \tag{A7.4}$$

The Gaussian integrals in (A7.4) are evaluated as in appendix 5 to yield (in the $L \to \infty$, $m_0 \to 0$ limit)

$$C_n^{(2)}(\boldsymbol{R}) = (C_n^0(\boldsymbol{R})/2)a^{\beta_0^2/2\pi} \int_{-\infty}^{\infty} d^2x^{(1)}\, d^2x^{(2)} [g(x^{(1)}, x^{(2)}, \boldsymbol{R}) - 1]$$
$$\times [(x^{(1)} - x^{(2)})^2 + a^2]^{-\beta_0^2/4\pi}. \tag{A7.5}$$

where

$$g(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{R}) \equiv \left(\frac{(R - x^{(1)2} + a^2)(x^{(2)2} + a^2)}{(\boldsymbol{R} - \boldsymbol{x}^{(2)2} + a^2)(\boldsymbol{x}^{(1)2} + a^2)} \right)^{n\beta_0^2/4\pi}. \tag{A7.6}$$

Treatment of the integral in (A7.5) parallels that of (A6.1), the only difference being that there is an additional source of divergence, namely $x^{(1)} < \Delta$ and $|R - x^{(2)}| < \Delta$. These are power divergences, which are eliminated by the coupling to operators with lower n (Brezin *et al* 1976; § III.4.b). This region can therefore be ignored. The integral is IR convergent, the change of variable $\boldsymbol{x}^{(1)} \to \boldsymbol{w} \equiv \boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}$ can be affected and the replacements $\beta_0^2 = \beta^2 = 8\pi$ and $\alpha_0 = \alpha$ are appropriate, and the only UV singularities to be considered occur for $|\boldsymbol{w}| < \Delta$ with Δ infinitesimal. Analysis by regions, just as in appendix 6, shows that the ln R ln a terms from the integral cancel against similar terms coming from C_n^0. Hence $C_n(\boldsymbol{R})$ is multiplicatively renormalisable; the renormalisation constant is $Z_n \sim (\kappa^2a^2)^{-n^2\beta^2/4\pi}(1 + O(\alpha^2))$ whereupon equation (11.3) then follows.

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